$\mathbb{R}$-trees in topology, geometry, and group theory

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Contents

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1 Introduction

This paper is intended as a survey of the theory and applications of real trees from a topologist's point of view. The idea of an all-inclusive historical account was quickly abandoned at the start of this undertaking, but I hope to describe the main ideas in the subject with emphasis on applications outside the theory of $\mathbb{R}$-trees. The "Rips machine", i.e. the classification of measured laminations on 2-complexes, is the key ingredient. Roughly speaking, the Rips machine is an algorithm that takes as input a finite 2-complex equipped with a transversely measured lamination (more precisely, a band complex), and puts it in a "normal form". This normal form is surprisingly simple – the lamination is the disjoint union of finitely many sub-laminations each of which belongs to one of four types:

• *simplicial*: all leaves are compact and the lamination is a bundle over a leaf with compact 0-dimensional fiber,

• *surface*: geodesic lamination on a compact hyperbolic surface (or a cone-type orbifold),

• *toral*: start with a standard lamination of the n-torus by irrational planes of codimension 1 and restrict to the 2-skeleton; more generally, replace the torus by a cone-type orbifold covered by a torus (with the deck group leaving the lamination invariant),

• *thin*: this type is most interesting of all. It was discovered and studied by G. Levitt [Lev93b]. See section 5.3 for the definition and basic properties.

Measured laminations on 2-complexes arise in the study of $\mathbb{R}$-trees via a process called *resolution*. In the simplicial case, this idea goes back to J. Stallings and was used with great success by M. Dunwoody. If G is a finitely presented group that acts by isometries on an $\mathbb{R}$-tree, one wants to deduce the structure of G, given the knowledge of vertex and arc stabilizers. Bass-Serre theory [Ser80] solves this beautifully in the case of simplicial trees. For an exposition of Bass-Serre theory from a topological point of view, see [SW79].

I hope to convince the reader that the development of the theory of $\mathbb{R}$-trees is not an idle exercise in generalizations – indeed, in addition to the intrinsic beauty of the theory, $\mathbb{R}$-trees appear in "real life" as a brief look at the final section of this survey reveals. The reason for this is the construction presented in section 3, which takes a sequence of isometric actions of G on "negatively curved spaces" and produces an isometric action of G on an $\mathbb{R}$-tree in the (Gromov-Hausdorff) limit.

The central part of the paper (sections 4-6) is devoted to a study of the Rips machine and the structure theory of groups that act isometrically on $\mathbb{R}$-trees. The approach follows closely [BF95], and the reader is referred to that paper for more details. Gaboriau, Levitt, and Paulin have developed a different (but equivalent) point of view in a series of papers (see references, and in particular the survey [Pau97b] which puts everything together). For the historical developments and the state of the theory preceding Rips' breakthrough, see the surveys [Sha87] and [Sha91].

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2 Definition and first examples of $\mathbb{R}$-trees

Definition 2.1. Let (X, d) be a metric space and let $x, y \in X$. An *arc from x to y* is the image of a topological embedding $\alpha : [a, b] \to X$ of a closed interval $[a, b]$ (and we allow the possibility $a = b$) such that $\alpha(a) = x$ and

$\alpha(b) = y$. A *geodesic segment* from x to y is the image of an isometric embedding $\alpha : [a, b] \to X$ with $\alpha(a) = x$ and $\alpha(b) = y$.

Definition 2.2. We say that (X, d) is an $\mathbb{R}$-*tree* if for any $x, y \in X$ there is a unique arc from x to y and this arc is a geodesic segment.

Example 2.3. Let X be a connected 1-dimensional simplicial complex that contains no circles. For every edge e of X choose an embedding $e \to \mathbb{R}$. If $x, y \in X$, there is a unique arc A from x to y. This arc can be subdivided into subarcs $A_1, A_2, \cdots, A_n$ each of which is contained in an edge of X. Define the length of A_i as the length of its image in $\mathbb{R}$ under the chosen embedding, and define $d(x, y)$ as the sum of the lengths of the A_i's. The metric space (X, d) is an $\mathbb{R}$-tree. We say that an $\mathbb{R}$-tree is *simplicial* if it arises in this fashion.

Example 2.4. (SNCF metric) Take $X = \mathbb{R}^2$ and let e denote the Euclidean distance on X. Define a new distance d as follows. We imagine that there is a train line operating along each ray from the origin (=Paris). If two points $x, y \in X$ lie on the same ray, then $d(x, y) = e(x, y)$. In all other cases the train ride from x to y goes through the origin, so $d(x, y) = e(0, x) + e(0, y)$. The metric space (X, d) is a (simplicial) $\mathbb{R}$-tree.

Example 2.5. A slight modification of the previous example yields a non-simplicial $\mathbb{R}$-tree. Take $X = \mathbb{R}^2$ and imagine trains operating on all vertical lines as well as along the x-axis. Thus $d(x, y) = e(x, y)$ when x, y are on the same vertical line, and $d(x, y) = |x_2| + |y_2| + |x_1 - y_1|$ otherwise, where we set $x = (x_1, x_2)$ and $y = (y_1, y_2)$.

$\mathbb{R}$-trees that arise in applications tend to be separable (as metric spaces), and in fact they are the union of countably many lines. Example 2.5 can be easily modified to yield an example of a separable non-simplicial $\mathbb{R}$-tree (restrict to the subset of X consisting of the x-axis and the points with rational x-coordinate).

2.1 Isometries of $\mathbb{R}$-trees

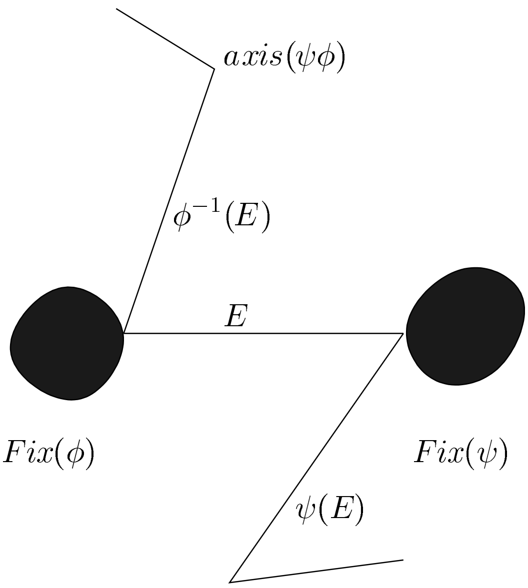
I will now recall basic facts about isometric actions on $\mathbb{R}$-trees. Proofs are a straightforward generalization from the case of simplicial trees that can be found in [Ser80]. Alternatively, the reader is referred to [MS84], [CM87], or [AB87].

Let $\phi : T \to T$ be an isometry of an $\mathbb{R}$-tree T. The *translation length* of ϕ is the number

$$\ell(\phi) = \inf\{d(x, \phi(x)) | x \in T\}$$

where d denotes the metric on T. The infimum is always attained. If $\ell(\phi) > 0$ there is a unique ϕ-invariant line (=isometric image of $\mathbb{R}$), called the *axis* of ϕ, and the restriction of ϕ to this line is translation by $\ell(\phi)$. In this case ϕ is said to be *hyperbolic*. If $\ell(\phi) = 0$, then ϕ fixes a non-empty subtree of T and is said to be *elliptic*.

Exercise 2.6. Let ϕ and ψ be two isometries of an $\mathbb{R}$-tree T. If they are both elliptic with disjoint fixed point sets, then the composition $\psi\phi$ is hyperbolic, and $\ell(\psi\phi)$ is equal to twice the distance between $Fix(\phi)$ and $Fix(\psi)$ (see figure below). If both ϕ and ψ are hyperbolic and their axes are disjoint, then $\psi\phi$ is hyperbolic, the translation length is equal to the sum of the translation lengths of ϕ and ψ plus twice the distance between the axes of ϕ and ψ, and the axis of $\psi\phi$ intersects both the axis of ϕ and of ψ.



Exercise 2.7. If $\{T_i\}_{i \in I}$ is a finite collection of subtrees of an $\mathbb{R}$-tree T such that all pairwise intersections are non-empty, then the intersection of the whole collection is non-empty.

Now let G be a group acting by isometries on an $\mathbb{R}$-tree T. The action is *non-trivial* if no point of T is fixed by the whole group. It is *minimal* if there is no proper G-invariant subtree.

Exercise 2.8. Use Exercise 2.7 to show that whenever a finitely generated group acts non-trivially on an $\mathbb{R}$-tree, then some elements of the group are mapped to hyperbolic isometries. Construct a (simplicial) counterexample to this statement when "finitely generated" is omitted from the hypotheses.

Proposition 2.9. *Assume that G is finitely generated and that the action of G on T is non-trivial. Then T contains a unique G-invariant subtree $T' \subset T$ such that the action restricted to T' is minimal. Further, T' is the union of at most countably many lines.*

Proof. Let T' be the union of the axes of hyperbolic elements in T. The only fact that needs a proof is that T' is non-empty and connected, and this follows from the exercises above. $\square$

We will often replace a given $\mathbb{R}$-tree with the minimal subtree without saying so explicitly.

2.2 δ-hyperbolic spaces – a review

We recall the notion of "negative curvature" for metric spaces, due to M. Gromov [Gro87]. For an inspired exposition see [GdlH90].

Definition 2.10. Let (X, d) be a metric space and $* \in X$ a basepoint. For $x, y \in X$ define $(x \cdot y) = \frac{1}{2}(d(*, x) + d(*, y) - d(x, y))$. For $\delta \geq 0$ we say that $(X, *, d)$ is *δ-hyperbolic* if for all $x, y, z \in X$ we have

$$(x \cdot y) \geq \min((x \cdot z), (y \cdot z)) - \delta.$$

Example 2.11. Let X be an $\mathbb{R}$-tree. Then $(x \cdot y)$ equals the distance between $*$ and the segment $[x, y]$. Further, if $x, y, z \in X$, then $(x \cdot y) \geq \min((x \cdot z), (y \cdot z))$. Thus $\mathbb{R}$-trees are 0-hyperbolic spaces. The converse is given in the next lemma.

Example 2.12. Hyperbolic space $\mathbb{H}^n$ and any complete simply-connected Riemannian manifold with sectional curvature $\leq -\epsilon < 0$ is δ-hyperbolic for some $\delta = \delta(\epsilon)$.

If $(X, *, d)$ is δ-hyperbolic and if $*'$ is another basepoint, then $(X, *', d)$ is 2δ-hyperbolic. It therefore follows that the notions of "0-hyperbolic" and "hyperbolic" (i.e. δ-hyperbolic for some δ) don't depend on the choice of the basepoint.

A finitely generated group G is *word-hyperbolic* [Gro87] if the word metric on G with respect to a finite generating set is hyperbolic. This notion is independent of the choice of the generating set.

The classification of isometries of hyperbolic spaces is more subtle than in the case of trees. Let $\phi : X \to X$ be an isometry of a hyperbolic metric space. The *translation length* can be defined as the limit

$$\ell(\phi) = \lim_{i\to\infty} \frac{1}{i} \inf_{x\in X} d(x, \phi^i(x)).$$

For a reasonable classification into hyperbolic, elliptic, and parabolic isometries it is necessary to assume something about X, e.g. that it is a geodesic metric space (any two points can be joined by a geodesic segment), or perhaps something weaker that guarantees that X does not have big holes. We will only need the following special case. If G is a word-hyperbolic group and $g \in G$ an element of infinite order, then left translation $t_g : G \to G$ by g has an *axis*, namely the set of points in G moved a distance $\leq \ell(t_g) + 10\delta$. ($G$ is δ-hyperbolic.) This set is quasi-isometric to the line.

2.3 "Connecting the dots" lemma

Lemma 2.13. *Let $(X, *, d)$ be a 0-hyperbolic metric space. Then there exists an $\mathbb{R}$-tree (T, d_T) and an isometric embedding $i : X \to T$ such that*

1. *no proper subtree of T contains $i(X)$, and*
2. *if $j : X \to T'$ is an isometric embedding of X into an $\mathbb{R}$-tree T', then there is a unique isometric embedding $k : T \to T'$ such that $ki = j$.*

In particular, T is unique up to isometry. Further, if a group G acts by isometries on X, then the action extends to an isometric action on T.

Proof. If $i : X \to T$ is an isometric embedding as in (1), then T is the union of segments of the form $I_x = [i(*), i(x)]$ for $x \in X$. The length of I_x is equal to $d(*, x)$, and two such segments I_x and I_y overlap in a segment of length $(x \cdot y)$. This suggest the construction of T. Start with the collection of segments $I_x = [0, d(*, x)]$ for $x \in X$ and then identify I_x and I_y along $[0, (x \cdot y)]$. For details, see e.g. [Ota96]. $\square$

3 How do $\mathbb{R}$-trees arise?

Isometric actions of a group G on an $\mathbb{R}$-tree arise most often as the Gromov-Hausdorff limits of a sequence of isometric actions of G on a negatively curved space X. The construction is due independently to F. Paulin [Pau88] and to M. Bestvina [Bes88]. See also the expository article [BS94].

We will make the formal definition in terms of the projectivized space of equivariant pseudometrics on G.

3.1 Convergence of based G-spaces

Let G be a discrete group. By a *G-space* we mean a pair (X, ρ) where X is a metric space and $\rho : G \to Isom(X)$ is a homomorphism (an *action*) to the group of isometries of X. A *based G-space* is a triple $(X, *, \rho)$ where (X, ρ) is a G-space and $*$ is a basepoint in X that is not fixed by every element of G.

Recall that a *pseudometric* on G is a function $d : G \times G \to [0, \infty)$ that is symmetric, vanishes on the diagonal, and satisfies the triangle inequality. Let $\mathcal{D}$ denote the space of all pseudometrics ("distance functions") on G that are not identically 0, equipped with compact-open topology. We let G act on $G \times G$ diagonally, and on $[0, \infty)$ trivially, and consider the subspace $\mathcal{ED} \subset \mathcal{D}$ of G-equivariant pseudometrics. Scaling induces a free action of $\mathbb{R}^+$ on $\mathcal{ED}$, and we denote by $\mathcal{PED}$ the quotient space, i.e. the space of *projectivized equivariant distance functions* on G. A pseudometric on G is δ-hyperbolic if the associated metric space is δ-hyperbolic (the class of the identity element is taken to be the basepoint).

A based G-space $(X, *, \rho)$ induces an equivariant pseudometric $d = d_{(X,*,\rho)}$ on G by setting

$$d(g, h) = d_X(\rho(g)(*), \rho(h)(*))$$

where d_X denotes the distance function in X. If the stabilizer under ρ of $*$ is trivial, then G can be identified with the orbit of $*$ via $g \leftrightarrow \rho(g)(*)$, and $d_{(X,*,\rho)}$ is the distance induced by d_X. We work with pseudometrics to allow for the possibility that distinct elements of G correspond to the same point of X.

Definition 3.1. We say that a sequence $(X_i, *_i, \rho_i)$, $i = 1, 2, 3, \cdots$ of based G-spaces *converges* to the based G-space $(X, *, \rho)$ and write

$$\lim_{i\to\infty} (X_i, *_i, \rho_i) = (X, *, \rho)$$

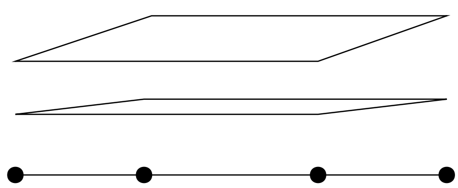
provided $[d_{(X_i,*_i,\rho_i)}] \to [d_{(X,*,\rho)}]$ in $\mathcal{PED}$.

3.2 Example: Flat tori

To illustrate this, let us take $G = \mathbb{Z} \times \mathbb{Z}$ and $X = E^2$, the Euclidean plane. We will obtain actions of G on the real line as limits of discrete actions of $\mathbb{Z} \times \mathbb{Z}$ on E^2. The group $\mathbb{Z} \times \mathbb{Z}$ can act on E^2 by isometries in many different ways. We will only consider discrete orientation-preserving isometric actions, and those consist necessarily of translations and form the universal covering group of a flat 2-torus. Two such actions of $\mathbb{Z} \times \mathbb{Z}$ will be considered equivalent if there is a similarity of E^2 conjugating one action to the other, i.e. if the corresponding (marked) tori are conformally equivalent. It is convenient to identify the group of translations of E^2 with $\mathbb{C}$. Thus two actions $\rho_1, \rho_2 : G \to \mathbb{C}$ are equivalent if there is a complex number α such that $\rho_2(g) = \alpha\rho_1(g)$ for all $g \in \mathbb{Z} \times \mathbb{Z}$ or $\rho_2(g) = \alpha\overline{\rho_1(g)}$ for all $g \in \mathbb{Z} \times \mathbb{Z}$. Each equivalence class $[\rho]$ is uniquely determined by the complex-conjugate pair $\{z, \overline{z}\}$ where $z = \frac{\rho(0,1)}{\rho(1,0)}$, and thus the set of all equivalence classes can be identified with the upper half-plane $\{z \in \mathbb{C} | Im(z) > 0\}$.

Let us take a sequence $\{z_n\}$ of points in the upper half-plane and let ρ_n be a representative of the equivalence class determined by z_n. Fix a basepoint $*$ in E^2.

Proposition 3.2. *Suppose that $z_n \to r \in \mathbb{R} \cup \{\infty\}$. Then the sequence $(E^2, *, \rho_n)$ converges to the (unique up to scale) based G-space $(\mathbb{R}, 0, \rho)$ where the action ρ consists of translations and $\frac{\rho(0,1)}{\rho(1,0)} = r$.*



The fundamental domain degenerates to a segment.

Proof. Suppose for concreteness that $r \in \mathbb{R}$. We take ρ_n so that $\rho_n(1,0) = 1$ and $\rho_n(0,1) = z_n$. Then $\rho_n(g) \to \rho(g) \in \mathbb{R} \subset \mathbb{C}$ and the claim follows. $\square$

Thus this construction recovers the usual compactification of the upper half-plane by the circle $\mathbb{R} \cup \{\infty\}$.

If in this example we replace E^2 by E^n and $\mathbb{Z} \times \mathbb{Z}$ by $\mathbb{Z}^n$, the same construction would produce actions of $\mathbb{Z}^n$ by translations on E^m, $0 < m < n$ and would provide an equivariant compactification of the symmetric space $SL_n(\mathbb{R})/SO_n(\mathbb{R})$.

3.3 $\mathbb{R}$-trees as limits of based δ-hyperbolic G-spaces

The main reason for interest in $\mathbb{R}$-trees is the following result. Note that if a sequence of based G-spaces converges, the limit is far from being unique. In special situations we can take the limit to be an $\mathbb{R}$-tree.

Theorem 3.3. *Let $(X_i, *_i, \rho_i)$ be a convergent sequence of based G-spaces. Assume that*

1. *there exists $\delta \geq 0$ such that every X_i is δ-hyperbolic, and*
2. *there exists $g \in G$ such that the sequence $d_{X_i}(*, \rho_i(g)(*))$ is unbounded.*

Then there is a based G-tree $(T, *)$ and an isometric action $\rho : G \to Isom(T)$ such that $(X_i, *_i, \rho_i) \to (T, *, \rho)$.

Proof. The limiting pseudo-metric d on G is 0-hyperbolic, as it is the limit of pseudo-metrics $\frac{d_{(X_i, *_i, \rho_i)}}{d_i}$ with $d_i \to \infty$ (by (2)) and the ith pseudo-metric is $\frac{\delta}{d_i}$-hyperbolic (by (1)). Now apply the connecting-the-dots Lemma 2.13 to the induced metric space. $\square$

Exercise 3.4. Let $G = \mathbb{Z}$, $X_i = \mathbb{H}^2$ (hyperbolic plane), and the representation ρ_i sends the generator $1 \in \mathbb{Z}$ to a hyperbolic isometry whose translation length is 1 and whose axis passes at distance i from the basepoint in $\mathbb{H}^2$. Show that the limiting $\mathbb{R}$-tree T can be identified with the cone on $\mathbb{Z}$ and the limiting action ρ is the translation action on $\mathbb{Z}$ coned off. The basepoint is a point in $\mathbb{Z}$.

Example 3.5. Let $f : F_n \to F_n$ be an automorphism of the free group $G = F_n =< x_1, \cdots, x_n >$ of rank n that sends each basis element to a "positive word", i.e. a product of basis elements (not involving their inverses). Suppose that $\lambda > 0$ is the unique eigenvalue of the abelianization of f, viewed as an automorphism of $\mathbb{Z}^n$, with a corresponding eigenvector with non-negative coordinates $a_1, \cdots, a_n$. For X_i take F_n with the word metric, the basepoint is 1, and let ρ_i be the representation that sends $g \in F_n$ to the left translation by $f^i(g)$. The scaling factor can be taken to be $d_i = \lambda^i$. If $\lambda > 1$ we obtain in the limit an action ρ of F_n on an $\mathbb{R}$-tree. The positivity requirement was imposed to ensure that for some g the sequence of lengths of $f^i(g)$ grows at the "top speed", i.e. as $(const)\lambda^i$. The limiting tree can be described quite explicitly. For example, if g is a positive word, then the distance between the basepoint $* \in T$ and its image under $\rho(g)$ is $k_1a_1 + \cdots + k_na_n$, where k_j is the number of times the generator x_j appears in the word g.

More generally, this construction can be performed with the train-track maps of [BH92]. With the right choice of a train-track map one obtains free nonsimplicial isometric actions of the free group F_n $(n > 2)$ on $\mathbb{R}$-trees (see [Sha91]).

3.4 Finding approximate subtrees

We now assume that we are in the situation of Theorem 3.3 and we examine the limiting tree in more detail. Thus we assume that there is a sequence $d_i \to \infty$ such that

$$d_T(\rho(g)(*), \rho(h)(*)) = \lim_{i\to\infty} \frac{d_{X_i}(\rho_i(g)(*_i), \rho_i(h)(*_i))}{d_i}.$$

If x is a point in T that belongs to the orbit of $*$, then x can be "approximated" by the corresponding point $x_i \in X_i$ in the orbit of $*_i$. If two points in the orbit of $*$ coincide, then the corresponding points in X_i are "close" (more precisely, the distance between them divided by d_i goes to 0 as $i \to \infty$).

We now extend this discussion to all points of T. We will assume in addition that each X_i is a *geodesic metric space*, i.e. that any two points $x, y \in X_i$ are joined by a geodesic segment.

Let $x \in T$ be an arbitrary point. Fix a finite subset $F \subset G$ and define a set $X_i(F, x) \subset X_i$ "approximating" x to be the set of all points $x_i \in X_i$ that can be constructed as follows. Choose $g, h \in F$ so that x is on the segment in T connecting $\rho(g)(*)$ and $\rho(h)(*)$ and choose a geodesic segment in X_i connecting $\rho_i(g)(*_i)$ and $\rho_i(h)(*_i)$, and let x_i be the point on this segment that divides it in the same ratio as the point x divides the segment $[\rho(g)(*), \rho(h)(*)]$.

Of course, it might happen that $X_i(F, x) = \emptyset$ if there are no $g, h \in F$ as above. The following proposition summarizes the basic properties of this construction:

Proposition 3.6.

1. *equivariance:* $X_i(gF, \rho(g)x) = \rho_i(g)X_i(F, x)$.
2. *monotonicity: If* $F \subset F'$ *then* $X_i(F, x) \subset X_i(F', x)$.
3. *small diameter:* $\frac{1}{d_i}diam\ X_i(F, x) \to 0$ *as* $i \to \infty$.
4. *metric convergence: Let* $x, y \in T$. *Then for all finite* $F \subset G$ *and all choices* $x_i \in X_i(F, x)$, $y_i \in X_i(F, y)$ *we have*
$$\frac{1}{d_i} d_{X_i}(x_i, y_i) \to d_T(x, y).$$
5. *non-triviality: For every* $x \in T$ *there is a 2-element set* $F \subset G$ *such that* $X_i(F, x) \neq \emptyset$ *for all* i

Proof. Items 1 and 2 follow directly from the definition. Item 3 is an exercise in δ-hyperbolic geometry. Item 4 also follows directly from definitions if $F = \{g, h\}$ so that the segment in T joining $\rho(g)(*)$ and $\rho(h)(*)$ contains both x and y (the existence of such g, h follows from item 1 of Lemma 2.13, which also implies item 5). The general case then follows from 2 and 3 by enlarging F. □

3.5 Selecting the basepoint and the Compactness Theorem

We now assume that an action $\rho : G \to Isom(X)$ is given, and we consider the problem of locating a "most centrally located point" for this action. We will then use this point as the basepoint. In Example 3.4 the basepoint $*_i$ should be chosen on the axis of $\rho_i(1)$, and then the limiting action would be nontrivial.

The problem of finding a good basepoint has a satisfactory solution when G is finitely generated and X is a *proper* δ-hyperbolic metric space, and this is what we assume from now on. (A metric space is *proper* if closed metric balls are compact.) We also fix a finite generating set $S \subset G$. Let $F = F_{S,\rho} : X \to [0, \infty)$ be the function defined by

$$F(x) = \max_{g \in S} d_X(x, \rho(g)(x)).$$

The following lemma is an exercise.

Lemma 3.7. *Assume that $\rho : G \to Isom(X)$ is non-elementary (i.e. it does not fix a point at infinity). Then $F : X \to [0, \infty)$ is a proper map. In particular, F attains its global minimum.* $\square$

We call a point $x \in X$ *centrally located (with respect to the action $\rho : G \to Isom(X)$ and the generating set S)* if F attains its global minimum at x.

Proposition 3.8. *Suppose that under the hypotheses of Theorem 3.3 each X_i is proper and that the basepoints $*_i$ are centrally located (with respect to ρ_i and a fixed finite generating set S for G). Then the limiting action $\rho : G \to Isom(T)$ does not have global fixed points.*

Proof. We can take $d_i = \max_{g \in S} d_{X_i}(*_i, \rho_i(g)(*_i))$. Suppose $x \in T$ is a global fixed point. Choose a finite subset $F \subset G$ so that $X_i(F, x) \neq \emptyset$. We will argue that for any $x_i \in X_i(F, x)$ and any $g \in S$ we have $\frac{1}{d_i} d_{X_i}(x_i, \rho_i(g)(x_i)) \to 0$ as $i \to \infty$, contradicting (for large i) the assumption that $*_i$ is centrally located. Indeed, $\rho(g)(x) = x$ coupled with equivariance property implies $X_i(gF, x) = \rho_i(g) X_i(F, x)$, so by monotonicity both x_i and $\rho_i(g)(x_i)$ belong to $X_i(gF \cup F, x)$, so the claim follows from the small diameter property. $\square$

Theorem 3.9 (Compactness Theorem). *Suppose that (X, d) is a proper δ-hyperbolic metric space and that $\rho_i : G \to Isom(X)$ a sequence of non-elementary representations of a finitely generated group G. Assume that the group $Isom(X)$ acts cocompactly on X, i.e. that there is a compact subset $K \subset X$ whose $Isom(X)$-translates cover X. Then one of the following holds, possibly after passing to a subsequence.*

1. *There exist isometries $\phi_i \in Isom(X)$ such that the sequence of conjugates $\rho_i^{\phi_i}$ converges in the compact-open topology to a representation $\rho : G \to Isom(X)$.*

2. *For each i there exists a centrally located point $x_i \in X$ for the representation ρ_i such that the sequence of based G-spaces (X, x_i, ρ_i) converges to an action of G on an $\mathbb{R}$-tree T without global fixed points.*

Proof. Let x_i be a centrally located point for ρ_i. If the sequence $d_i = \max_{g \in S} d_X(x_i, \rho_i(g)(x_i))$ converges to infinity, then item 2 holds, by the preceding proposition. Otherwise, after passing to a subsequence, the d_i's are uniformly bounded. In that case choose $\phi_i \in Isom(X)$ that sends x_i into $K \subset X$ and apply Arzela-Ascoli to the conjugates $\rho_i^{\phi_i}$ to see that item 1 holds in this case. $\square$

In the situation (2) of the Compactness Theorem, assuming that X is a geodesic metric space or a hyperbolic group G equipped with a word metric, it can also be argued that the translation length $\ell(\rho(g))$ is equal to the limit $\lim_{i\to\infty} \frac{\ell(\rho_i(g))}{d_i}$.

3.6 Arc stabilizers

We now investigate, in the situation (2) of the Compactness Theorem, the arc stabilizers in the limiting action $\rho : G \to Isom(T)$. We restrict ourselves to two frequently encountered settings, when the arc stabilizers turn out to be "elementary".

Many reasonable groups, such as linear groups, satisfy the so called "Tits Alternative". This means that their subgroups are either "small" (virtually solvable) or "large" (contain a nonabelian free group). Word-hyperbolic groups satisfy a strong form of the Tits alternative: any subgroup either contains a nonabelian free group or it is virtually cyclic (elementary). Accordingly, an action of a group on an $\mathbb{R}$-tree is said to be *small* if it is non-trivial (there are no global fixed points), minimal, and all arc stabilizers are small.

Proposition 3.10. *Let $H \subset G$ be the stabilizer under ρ of a non-degenerate arc in T.*

1. *If each X_i is a copy of the Cayley graph Γ of a word hyperbolic group with respect to a fixed finite generating set, and each $\rho_i : G \to Isom(\Gamma)$ is a free action whose image is contained in the subgroup consisting of left translations, then H is virtually cyclic.*

2. *If each X_i is a copy of a fixed rank 1 symmetric space $\mathbb{H}$ (real, complex, quaternionic hyperbolic space, or the Cayley plane), and each ρ_i is discrete and faithful, then H is virtually nilpotent.*

Proof. Let $[a, b] \subset T$ be a non-degenerate segment fixed by H (under the action by ρ). Choose a sufficiently large finite subset $F \subset G$ and points $a_i \in X_i(F, a)$ and $b_i \in X_i(F, b)$. Let c_i be the midpoint on a geodesic segment σ_i connecting a_i and b_i.

(1) Say Γ is δ-hyperbolic. The key claim is that if $h, k \in H$ then, for large i, the left translation $\rho_i([h, k])$ moves c_i to a point at distance $< 20\delta$ from c_i. There is an upper bound to the number of left translations of Γ that move a given point a distance $\leq 20\delta$. Since the commutators $[h, k]$ for

$h, k \in H$ generate the commutator subgroup $[H, H]$ of H, it follows from the freeness assumption that $[H, H]$ is finitely generated and in particular H is not a nonabelian free group. Since the same argument can be applied to any subgroup of H, we conclude that H does not contain a nonabelian free group, and hence it is virtually cyclic.

The idea of proof of the above key claim is that $\rho_i(h)$ and $\rho_i(k)$ map σ_i to a geodesic segment whose endpoints are within $\frac{1}{100} length(\sigma_i)$ of the endpoints of σ_i, and so these segments, except near the endpoints, run within 2δ of σ_i, i.e. $\rho_i(h)$ and $\rho_i(k)$ can be thought of (modulo small error) as translating along σ_i. Consequently, the commutator $\rho_i([h, k])$ fixes σ_i (modulo small error and away from the endpoints). Details are in [Bes88] and [Pau88].

(2) The proof here is a modification of (1), plus the Margulis lemma. Let μ be the Margulis constant for $\mathbb{H}$, so that if a discrete group of isometries of $\mathbb{H}$ is generated by isometries that move a point $x_0 \in \mathbb{H}$ a distance $< \mu$, then the group is virtually nilpotent. Arguing as in the key claim above, one can show that if $h, k \in H$, then for large i the isometry $\rho_i([h, k])$ moves c_i a distance $< \mu$. It then follows that every finitely generated subgroup of $[H, H]$ is virtually nilpotent and so $\rho_i(H)$ must be elementary (i.e. virtually nilpotent). □

3.7 Stable actions

Definition 3.11. Suppose a group G is acting isometrically on an $\mathbb{R}$-tree T. A subtree of T is *non-degenerate* if it contains more than one point. A non-degenerate subtree $T_1 \subset T$ is said to be *stable* (with respect to the action) if for every non-degenerate subtree $T_2 \subset T_1$ we have the equality $Fix(T_1) = Fix(T_2)$ of pointwise stabilizers. The group action on T is *stable* if it is non-trivial, minimal, and every non-degenerate tree in T contains a stable subtree.

Group actions that tend to arise in practice are stable. For example, small actions of hyperbolic groups are stable. More generally, if the collection of arc stabilizers satisfies the ascending chain condition, then the (non-trivial and minimal) group action is stable. Note that if two stable subtrees of T have a non-degenerate intersection, then their union is a stable subtree. In particular, each stable subtree is contained in a unique maximal stable subtree.

The study of stable actions quickly reduces to the study of actions with trivial arc stabilizers (see Corollary 5.9 of [BF95]). To see the idea, assume that T is covered by maximal stable subtrees $\{T_i\}_{i \in I}$. Note that $T_i \cap T_j$ is at most a point for $i \neq j$. Now construct a simplicial tree S as follows. There are two kinds of vertices in S. There is a vertex for each maximal stable subtree T_i, and there is a vertex for each point of T that equals the intersection of distinct maximal stable subtrees. An edge is drawn from a vertex v of the first kind, determined by T_i, to the vertex w of the second kind, determined by $x \in T$, precisely when $x \in T_i$. The group G acts simplicially, without inversions of edges, on S. The stabilizer $Fix_S([v, w])$ of the edge $[v, w]$ described above fixes a point of T and the underlying assumption is that we understand arc and point stabilizers in T. We then appeal to Bass-Serre theory [Ser80] to conclude that either G splits over an edge stabilizer in S or that G fixes a vertex of S. In the latter case, in view of nontriviality and minimality of the action of G on T, it follows that T itself is a stable tree, so after factoring out the kernel of the action, the induced action has trivial arc stabilizers.

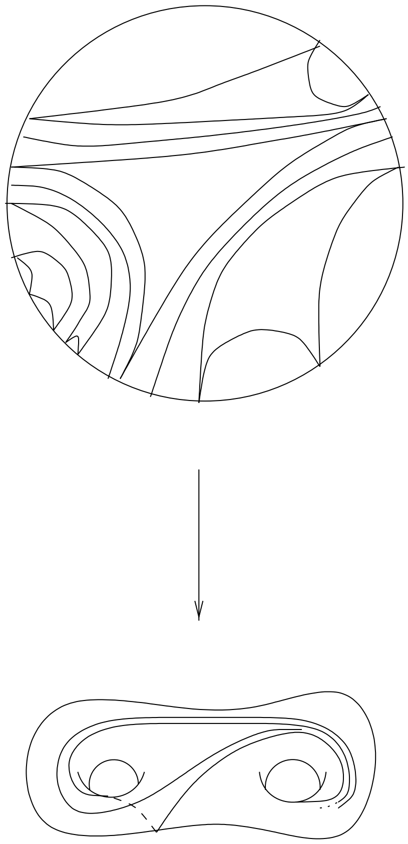
4 Measured laminations on 2-complexes

We now review the basics of measured laminations. For more information and details the reader is referred to [MS88a].

Definition 4.1. A closed subset Λ of a locally path-connected metrizable space X is a *lamination* if every point $x \in \Lambda$ has a neighborhood U such that the pair $(U, U \cap \Lambda)$ is homeomorphic to the pair $(V \times (0,1), V \times C)$ for some topological space V and some compact totally disconnected subset $C \subset (0,1)$. Such a homeomorphism is called a *chart*. The path components of Λ are called *leaves*.

If X is a closed manifold, any codimension 1 (locally flat) submanifold is a lamination. More typically, the set C in the definition is the Cantor set.

Example 4.2. Let X be a closed hyperbolic surface. Let γ_i be a sequence of simple closed geodesics in X. After possibly passing to a subsequence, this sequence converges in the Hausdorff metric to a closed subset Λ of X. One can check [CB88] that Λ is a lamination, and that the leaves of Λ are simple geodesics (closed or biinfinite). Such Λ is called a *geodesic lamination.*



A geodesic lamination on a hyperbolic surface and its universal cover. In the cover, countably many leaves are sides of ideal polygons. All other leaves are inaccessible from the complement.

Definition 4.3. Let $\Lambda \subset X$ be a lamination and $\alpha : [a, b] \to X$ a path in X such that $\alpha(a), \alpha(b) \notin \Lambda$. We say that α is *transverse* to Λ if for every $t \in [a, b]$ with $\alpha(t) \in \Lambda$ there is a chart $h : (U, U \cap \Lambda) \to (V \times (0,1), V \times C)$ at $\alpha(t) \in X$ such that the map $pr_{(0,1)}h\alpha$ is a local homeomorphism at t.

Definition 4.4. A *transverse measure* on a lamination $\Lambda \subset X$ is a function μ that assigns a nonnegative real number $\mu(\alpha)$ to every path α transverse to Λ and satisfies the following properties.

1. If α is the concatenation of paths β and γ both of which are transverse to Λ, then $\mu(\alpha) = \mu(\beta) + \mu(\gamma)$.

2. Every $x \in \Lambda$ has a chart $(U, U \cap \Lambda) \approx (V \times (0,1), V \times C)$ and there

is a Borel measure ν on $(0,1)$ supported on C such that for any path $\alpha : [a,b] \to U$ with endpoints outside Λ that projects 1-1 to an interval in $(0,1)$, the measure $\mu(\alpha)$ equals the ν-measure of the projection.

The number $\mu(\alpha)$ is the *measure* of α.

Exercise 4.5. If two paths are homotopic through paths transverse to Λ, then they have the same measure. If a path is reparametrized, its measure does not change.

The *support* of μ is the complement of the set of points such that $\mu(\alpha) = 0$ whenever the image of α is contained in a sufficiently small neighborhood of the point. A lamination is *measured* if it is equipped with a transverse measure.

The support of μ is always a sublamination of Λ. We say that Λ has *full support* if the support is all of Λ.

Example 4.6. Let Λ be a geodesic lamination on a hyperbolic surface X. If Λ is the finite union of simple closed curves, any transverse measure assigns a nonnegative real number, the *multiplicity* to each leaf, and the measure of any path transverse to Λ is the geometric intersection number with Λ, counted with multiplicity. Conversely, any such assignment determines a transverse measure. Now suppose that ℓ is an infinite leaf of Λ. We will construct a transverse measure on Λ, called the *counting* or the *hitting* measure. Triangulate the surface X so that the vertices are in the complement of Λ, all edges are geodesic segments, and each triangle is contained in a chart for Λ. For each edge e the intersection $e \cap \Lambda$ is totally disconnected. Choose a point in each component of $Int\, e \setminus \Lambda$. A transverse measure on Λ is determined by its values on the subintervals of the edges e with endpoints in the selected countable set. Conversely, if μ is defined on these countably many special intervals and the following two conditions hold, then μ extends uniquely to a transverse measure on Λ:

1. (additivity) If a special interval I is the concatenation of two special subintervals I_1 and I_2, then $\mu(I) = \mu(I_1) + \mu(I_2)$.

2. (compatibility) If I_1 and I_2 are special intervals belonging to two edges of the same triangle T in the triangulation, and if there is an embedded quadrilateral in T with two opposite sides I_1 and I_2, and the other two opposite sides disjoint from Λ, then $\mu(I_1) = \mu(I_2)$.

We will now construct a hitting measure on Λ. Choose a sequence of longer and longer closed subintervals $L_1, L_2, \cdots$ of the leaf ℓ. For a special interval I define

$$\mu_i(I) = \frac{N(L_i, I)}{N(L_i, X^{(1)})}$$

where $N(L_i, I)$ is the number of intersection points in $L_i \cap I$ and similarly $N_i = N(L_i, X^{(1)})$ is the number of intersection points between L_i and the 1-skeleton. Since ℓ is an infinite leaf, we have $N_i \to \infty$. Additivity and compatibility hold approximately for μ_i, i.e. in both cases the difference between the left-hand and the right-hand side is in the interval $[-\frac{1}{N_i}, \frac{1}{N_i}]$. Using a diagonalization process, pass to a subsequence if necessary so that $\lim \mu_i(I)$ exists for each special interval I, and set the limiting value equal to $\mu(I)$. The support of μ is generally smaller than Λ.

Example 4.7. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a linear map that is injective when restricted to $\mathbb{Z}^n$ and consider the foliation of $\mathbb{R}^n$ by the level sets of f and the induced foliation $\mathcal{F}$ on the torus $T^n = \mathbb{R}^n/\mathbb{Z}^n$. The map f also defines a transverse measure on the two foliations. There is a standard way of converting the measured foliation $\mathcal{F}$ to a measured lamination Λ (and vice-versa); indeed, [BF95] is written in the language of foliations. More precisely, there is a map $T^n \to T^n$ whose point-preimages are arcs and points and the preimage of each leaf of $\mathcal{F}$ is either a leaf of Λ or the closure of a complementary component of Λ. This map is modeled on the Cantor function $[0,1] \to [0,1]$, which converts the foliation of $[0,1]$ by points to the lamination on $[0,1]$ whose underlying set is the Cantor set.

4.1 Sacksteder's Theorem

We say that two paths γ and δ transverse to a lamination Λ are *pushing equivalent* if, after possibly reparametrizing one, they are homotopic through paths transverse to Λ. If Λ is equipped with a transverse measure μ, then by Exercise 4.5 we have $\mu(\gamma) = \mu(\delta)$. In particular, a measured lamination of full support satisfies the following *non-nesting condition*:

If $\gamma : [a,b] \to X$ and $\delta : [c,d] \to X$ are pushing-equivalent and γ is a subpath of δ (i.e. $\gamma = \delta|[a,b]$), then $\delta([c,d] \setminus [a,b]) \subset X \setminus \Lambda$.

There is a remarkable converse, due to R. Sacksteder.

Theorem 4.8. *[Sac65] Suppose X is compact and $\Lambda \subset X$ is a lamination* *on X satisfying the above non-nesting condition. Then there is a non-trivial transverse measure on Λ, possibly not of full support.*

It is easy to construct examples of non-nesting laminations on compact spaces that do not support a transverse measure. For example, take a geodesic measured lamination on a hyperbolic surface and replace a noncompact leaf with a parallel family of leaves. For an $\mathbb{R}$-tree version of Sacksteder's theorem, see [Lev98].

4.2 Decomposition into minimal and simplicial components

We say that a lamination $\Lambda \subset X$ is *simplicial* if there is a leaf ℓ of Λ, a closed neighborhood N of Λ in X and a map $N \to \ell$ which is an I-bundle and whose restriction to Λ is a bundle map with 0-dimensional fibers. A lamination Λ is *minimal* if every leaf of Λ is dense in Λ. When the underlying space X is compact, the lamination that supports a transverse measure always decomposes into simplicial and minimal sub-laminations.

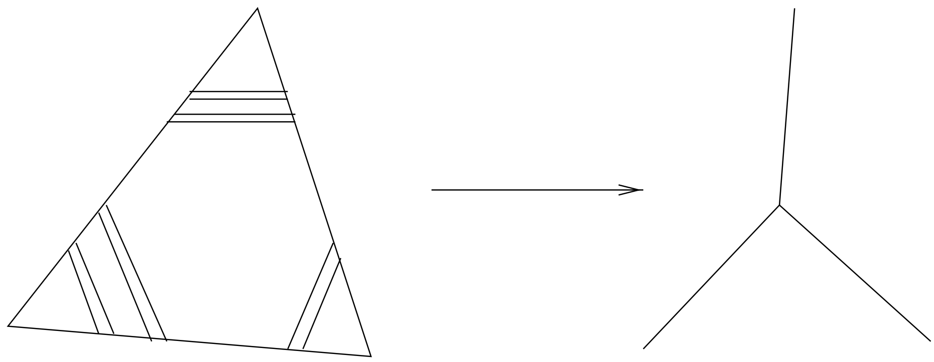
Theorem 4.9. *(Theorem 3.2 in [MS88a]) Let X be compact and $\Lambda \subset X$ a lamination that admits a transverse measure with full support. Then Λ is the disjoint union $\Lambda_1 \sqcup \Lambda_2 \sqcup \cdots \sqcup \Lambda_n$ with each Λ_i either simplicial or minimal.*

On a closed hyperbolic surface, imagine a lamination consisting of two closed geodesics and a biinfinite geodesic that spirals towards the closed geodesics, one in each direction. Such a lamination does not decompose into simplicial and minimal sub-laminations. It is also not hard to show directly that this lamination does not support a transverse measure; indeed, this lamination is not even non-nesting.

4.3 Resolutions

Let G be a finitely presented group, and assume that G is acting non-trivially and minimally on an $\mathbb{R}$-tree T (as usual, by isometries). Since G is finitely presented, there is a finite simplicial complex K of dimension ≤ 2 whose fundamental group is G. We now use T to construct a measured lamination Λ on K and an equivariant map $f : \tilde{K} \to T$ from the universal cover of K to T that sends leaves of the preimage lamination $\tilde{\Lambda} \subset \tilde{K}$ to points. We refer to this map as a *resolution*. In the case of simplicial trees this construction has been extensively used by M. Dunwoody (the leaves in this case are Dunwoody's "tracks").

To construct Λ and f, first choose a countable equivariant dense subset $D \subset T$ that includes all branch points of T ($v \in T$ is a branch point if the tripod, i.e. the cone on 3 points, can be embedded in T with the cone point mapped to v) and that intersects each arc in a dense set. This is possible by Proposition 2.9. Then define f on the vertices of $\tilde{K}$ so that the map is equivariant and sends each vertex into D. Next, extend f equivariantly to the edges of $\tilde{K}$. If the endpoints of a given edge e map to the same point under f, then define f on e to be the constant map. Otherwise, $f|e$ is chosen so that it is the Cantor function onto the arc whose boundary is $f(\partial e)$ with the preimage of each point in $D \cap f(e)$ an arc and the preimage of every other point in $f(e)$ a single point. The Cantor set of points in e that don't belong to the interior of a preimage arc is going to be the set $\tilde{\Lambda} \cap e$. Finally, extend f equivariantly to each 2-simplex σ of K so that for each $y \in D \cap f(\sigma)$ the preimage $f^{-1}(y) \cap \sigma$ is a convex triangle, quadrilateral, or a hexagon with vertices in $\partial\sigma$ and the preimage of every other point in $f(\sigma)$ is a straight line segment joining two distinct sides of σ.



These line segments are the components of $\tilde{\Lambda} \cap \sigma$. The transverse measure is defined by the requirement that if α is a path in $\tilde{K}$ that is transverse to $\tilde{\Lambda}$ and intersects each leaf at most once, then the measure of α is the distance in T between the f-images of the endpoints of α. This transverse measure is equivariant and descends to a transverse measure on the induced lamination $\Lambda \subset K$.

4.4 Dual trees

There is a construction that to a measured lamination Λ on a finite complex K assigns an $\mathbb{R}$-tree on which the fundamental group of the complex acts. Let $\tilde{K}$ be the universal cover of K and $\tilde{\Lambda}$ the induced lamination on $\tilde{K}$. Define a pseudometric $d : \tilde{K} \times \tilde{K} \to [0, \infty)$ by taking

$$d(x, y) = \inf_{\alpha} \tilde{\mu}(\alpha)$$

where $\tilde{\mu}$ is the induced transverse measure and the infimum runs over all paths that are transverse to $\tilde{\Lambda}$ and join x to y. It is not difficult to show that the associated metric space T is an $\mathbb{R}$-tree, called the *dual tree*, and that the deck group induces an isometric action of G on T. There is also the natural quotient map $f : \tilde{K} \to T$; it is equivariant and maps each leaf and each complementary component of $\tilde{\Lambda}$ to a point.

In general, many different leaves will map to the same point by f. For example, start with a geodesic lamination on a 4 times punctured sphere and then fill in the punctures. The dual tree in this case is a single point. It is reasonable to impose the condition that f restricted to each edge e of $\tilde{K}$ is the Cantor function that collapses precisely the closures of complementary components of $e \cap \Lambda$ in e. This condition is automatically satisfied when Λ arises as in the construction of a resolution. For the lack of a better term, we say that f is *locally injective* if it satisfies this condition.

Questions. Assume that f is locally injective. Is the infimum above always realized by a "minimizing" path α? Can f map distinct leaves of $\tilde{\Lambda}$ that do not belong to the closure of the same complementary component to the same point?

Example 4.10. Let Λ be a geodesic measured lamination on a closed hyperbolic surface such that the measure has full support and such that the complementary components are simply-connected (see e.g. [CB88]). The action of the fundamental group of the surface on the dual tree is free.

Notice that the constructions of a resolution and of the dual tree are generally not inverses of each other. For example, a free group admits many interesting non-simplicial actions on $\mathbb{R}$-trees (e.g. via the construction as in the preceding example applied to a punctured surface), while the dual of any resolution that uses a bouquet of circles for K is simplicial.

A resolution $f : \tilde{K} \to T$ is *exact* (see [BF95]) if all point preimages are connected. This is equivalent to the statement that each point preimage is either a leaf or the closure of a complementary component. A group action on an $\mathbb{R}$-tree T is *geometric* if it admits an exact resolution. For more information on geometric actions, see [LP97].

Frequently, one encounters the following situation: $f : \tilde{K} \to T$ is a resolution, $\tilde{\Lambda}$ the associated lamination, and $f' : \tilde{K} \to T'$ is the equivariant map to the tree dual to $\tilde{\Lambda}$. By construction, we have a factorization

$$f = \pi f'$$

for an equivariant map $\pi : T' \to T$. As remarked above, this map may not be an isometry. If f is an exact resolution, then π is an isometry.

Question. If π is an isometry, is f an exact resolution?

It is a consequence of the Rips machine that if the action on T is stable, then $f' : \tilde{K} \to T'$ is an exact resolution, so the potential pathologies in the questions above don't arise in the stable case.

In general, one can say that if f is not exact, then either π is not an isometry, or there exist two leaves of $\tilde{\Lambda}$ that can be joined by a path with arbitrarily small measure, but cannot be joined by a path of measure 0. In either case, there are two leaves of $\tilde{\Lambda}$ such that any path joining them has measure strictly larger than the distance between their f-images x and y. One can then construct a "better resolution" as follows. Choose a path in $\tilde{K}$ joining two such leaves. Attach a 2-cell $[0,1] \times [0,1]$ by gluing $[0,1] \times 0$ to the path. Map the other 3 boundary components to the arc $[x,y] \subset T$ (point if $x = y$). Then extend f to the 2-cell in the same way as when constructing a resolution. Finish the construction by attaching the whole orbit of 2-cells and extending to preserve equivariance. Slight care and subdivisions may be necessary to stay in the simplicial category. In the end, we have another resolution $f' : \tilde{K}' \to T$ and a factorization $f = f'\rho$, where $\rho : \tilde{K} \to \tilde{K}'$ is equivariant, sends leaves to leaves, and the images in $\tilde{K}'$ of the original pair of leaves are joined by a path whose measure is equal to the distance between x and y.

Continuing in this fashion, we can construct resolutions that more and more faithfully reflect the nature of T.

Proposition 4.11. *Assume that a finitely presented group G is acting by isometries on an $\mathbb{R}$-tree T and the action is non-trivial and minimal. For any finite collection $Y \subset T$ of points in T and any finite collection $G_0 \subset G$ of group elements there is a resolution $f : \tilde{K} \to T$ (K depends on Y and G_0)*

and a collection of points $Y' \subset \tilde{K}$ such that $\tilde{f}$ induces a bijection between Y' and Y and for any $a, b \in Y'$ and any $\gamma, \delta \in G_0$ there is a path α from $\gamma(a)$ to $\delta(b)$ whose measure is equal to $d(f(\gamma(a)), f(\delta(b)))$. $\square$

For example, if $x \in T$ and H is a finitely generated subgroup of the stabilizer $Stab(x)$, then we can construct a resolution $f : \tilde{K} \to T$ such that f sends a leaf or a complementary component D to x and $h(D) = D$ for all $h \in H$.

On the other hand, using the construction outlined in Example 3.5, one can show that there are examples of free actions of the free group F_3 such that every resolution is simplicial [BF].

4.5 Band complexes

It is more convenient to work with a special class of 2-complexes equipped with with measured laminations, called *band complexes*.

Definition 4.12. A *band* is the square $[0, 1] \times [0, 1]$ equipped with a measured lamination $C \times [0, 1]$ with measure of full support for a compact totally disconnected set $C \subset (0, 1)$.

A *multiinterval* Γ is the disjoint union of closed intervals equipped with a measured lamination $\Lambda(\Gamma)$ disjoint from the endpoints.

A *union of bands* is the space Y obtained from a multiinterval Γ by attaching a collection of bands. Each band $[0, 1] \times [0, 1]$ is attached via an embedding $\phi : [0, 1] \times \{0, 1\} \to \Gamma$ such that $\phi^{-1}(\Lambda(\Gamma)) = C \times \{0, 1\}$ and such that ϕ is measure-preserving. The measured lamination $\Lambda(\Gamma)$ pieces together with the measured laminations on the bands to produce a measured lamination $\Lambda(Y)$ on Y.

A *band complex* is the space X obtained from a union of bands Y by successively attaching 0-, 1-, and 2-cells (with PL attaching maps) so that

- There is a neighborhood of $\Lambda(Y)$ disjoint from the images of all attaching maps.
- The images of attaching maps of 1-cells are contained in Γ.

The band complex X is equipped with the induced measured lamination $\Lambda = \Lambda(X)$.

Example 4.13. Let X be the hyperbolic surface of Example 4.6. Each triangle in X intersects the lamination Λ in a collection of geodesic arcs, each spanning between two sides. Thus these arcs fall into at most 3 families according to which two sides they intersect. We can view X as a band complex as follows. The multi-interval Γ is obtained from the 1-skeleton by removing small disks around each vertex. Each triangle gives rise to at most 3 bands, one for each family of geodesic arcs. The vertices are the 0-cells, there are two 1-cells for each edge of the triangulation, connecting an endpoint to Γ. Finally, a triangle of the most interesting type (intersecting Λ in 3 families of arcs) gives rise to four 2-cells, three corner triangles, and a central hexagon. Simpler triangles give rise to fewer 2-cells.

Definition 4.14. Let X be a band complex and assume that $\pi_1(X)$ is acting on an $\mathbb{R}$-tree T. An equivariant map $f : \tilde{X} \to T$ is a *resolution* (or an *exact resolution*) if there is a triangulation of X so that f is a resolution (or an exact resolution) in the sense of section 4.3.

5 Rips machine

5.1 Moves on band complexes

Building on the work of Makanin [Mak83] and Razborov [Raz85], Rips has devised a "machine" that transforms any band complex into a "normal form". The reference for this section is [BF95]. Here we only outline some aspects of the Rips machine.

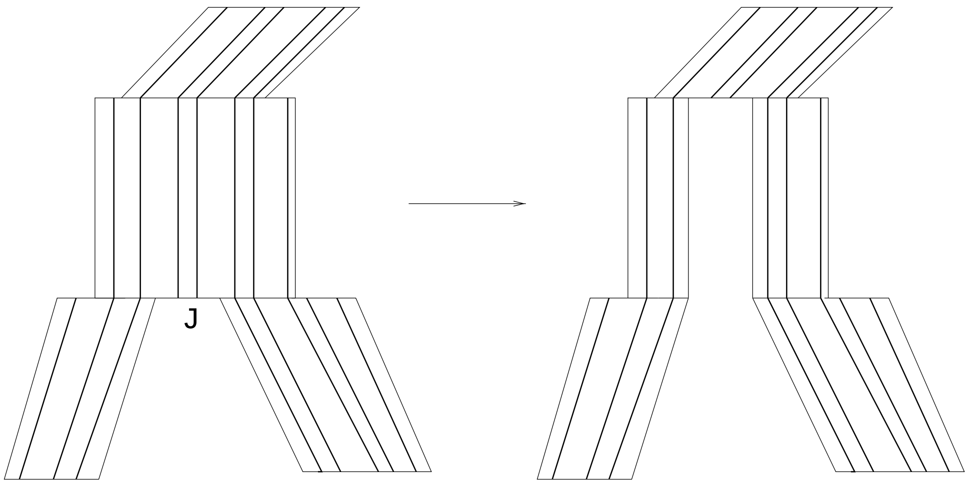
There is a list of 6 moves M0-M5 that can be applied to a band complex. The complete list is in section 6 of [BF95]. These moves are analogs of the elementary moves in simple homotopy theory, but they respect the underlying measured lamination. If a band complex X' is obtained from a band complex X by a sequence of these moves, then the following holds.

- There are maps $\phi : X \to X'$ and $\psi : X' \to X$ that induce an isomorphism between fundamental groups and preserve measure.
- If $f : \tilde{X} \to T$ is a resolution, then the composition $f\tilde{\psi} : \tilde{X}' \to T$ is also a resolution, and if $g : \tilde{X}' \to T$ is a resolution, then so is $g\tilde{\phi} : X \to T$.
- ϕ and ψ induce a 1-1 correspondence between the minimal components of the laminations on X and X'.

- $\tilde{\phi}$ and $\tilde{\psi}$ induce quasi-isometries between the leaves of the laminations in $\tilde{X}$ and $\tilde{X}'$.

By way of illustration, we describe one of the moves, namely (M5). An arc $J \subset \Gamma$ is said to be *free* if the endpoints of J are in the complement of Λ, J has positive measure, and it intersects only one attaching region of a band. A free subarc J is said to be a *maximal* free subarc if whenever $J' \supset J$ is a free subarc, then $J' \cap \Lambda = J \cap \Lambda$.

Assume that J is a maximal free subarc and that J is contained in the attaching region $[0,1] \times 0$ of a band $B = [0,1] \times [0,1]$. The move (M5) consists of collapsing $J \times [0,1]$ to $J \times 1 \cup Fr\ J \times [0,1]$. Typically, the band B will be replaced by two new bands, but if J contains one or both endpoints of the attaching region $[0,1]$, then B is replaced by 1 or 0 bands. Attaching maps of relative 1- and 2-cells whose images intersect *int* $J \times [0,1)$, can be naturally homotoped upwards.



5.2 The classification theorem

For simplicity, we will assume that X is a band complex and $f : \tilde{X} \to T$ is a resolution of an $\mathbb{R}$-tree T on which $G = \pi_1(X)$ is acting, and that the action has trivial arc stabilizers. This assumption is not necessary, but it dramatically simplifies the statements. The reason is that in this case it is always possible to remove annuli from a band complex. Imagine a band complex X that contains as a subcomplex an annulus $[0,1] \times S^1$ which is thought of as a single band with top and bottom attached to the same arc. If the measure of the arc $[0,1]\times p$ is positive, then the element of the fundamental group corresponding to the loop $q\times S^1$ fixes an arc in T and is therefore trivial. We can then collapse the annulus to the arc and replace X by the resulting complex X' (this is move (M1)).

We will also assume that $\pi_1(X)$ is torsion-free.

Theorem 5.1 (Rips,[BF95]). *Let X be a band complex such that $\tilde{X}$ resolves an action of the torsion-free group $\pi_1(X)$ on an $\mathbb{R}$-tree with trivial arc stabilizers. Then X can be transformed, using moves (M0-M5), to another band complex X' with the following properties. For each minimal component Λ_i' of the lamination Λ' on X' there is a subcomplex X_i' of X' that intersects Λ' in Λ_i' and these subcomplexes are pairwise disjoint. All inclusions $X_i'\hookrightarrow X'$ are π_1-injective, and so are all inclusions from a component of the frontier $Fr(X_i')$ into X_i'. Each X_i' is of one of the following 3 types:*

- *Surface type: X_i' is a compact surface with negative Euler characteristic and Λ_i' is a (filling) geodesic measured lamination (with respect to a hyperbolic structure on the surface). Each component of $Fr(X_i')$ is either a point or a boundary component of the surface.*
- *Toral type: X_i' is the 2-skeleton of the torus from Example 4.7 with the induced lamination. Each component of $Fr(X_i')$ is a point.*
- *Thin type: This type does not have a standard model. Its main feature is that it can be arranged that X_i' contains an arbitrarily thin band (i.e. with attaching regions of small measure) that intersects the rest of X' only in the two attaching regions. See more on this below.*

If $\pi_1(X)$ is not torsion-free, the theorem still holds provided that in the surface and toral types we allow for a finite number of cone-type orbifold points.

Traditionally, in terms of the dual tree (or the associated pseudogroup), surface type is called “interval exchange”, and toral type is called “axial”. Similarly, thin type is also called “Levitt type”, in honor of G. Levitt [Lev93b] who discovered and extensively studied this kind of a pseudogroup. Thin type has also been called “exotic”. We chose names that reflect the nature of the band complex, not the dual tree or the pseudogroup.

5.3 Thin type band complexes

We now describe band complexes of thin type in more detail. Suppose X is a band complex whose lamination Λ is minimal. If there are maximal free arcs, choose one and perform the collapsing move (M5) described above to obtain another band complex X_1. If X_1 contains maximal free arcs, choose one and collapse, etc. This process stops when there are no more free arcs. Note that a collapse might produce new free arcs. This process is called Process I in [BF95].

Definition 5.2. The band complex X is of *thin type* if it is equivalent under moves (M0-M5) to a complex for which the collapsing procedure never ends.

There is a concrete example of a thin band complexes in section 10 of [BF95]. This example has the additional feature that after each collapse the resulting band complex is a scaled down version of the original. R. Martin [Mar97] has studied the relation between the "periodicity" of the sequence of collapses and unique ergodicity of the underlying lamination. The band complexes associated to the "interesting" pseudogroups in [Lev93b] are thin.

If we focus on a single band in X, then under the collapsing process this band will get subdivided into more and more bands with arbitrarily small transverse measure (so these bands are thin, thus the name). In particular, eventually there will be bands whose interiors are disjoint from the attaching regions of the relative 2-cells. Such bands are called *naked bands.* A naked band induces a free product decomposition of $\pi_1(X)$ by cutting along an arc in the band that separates the two attaching regions. That this decomposition is non-trivial is the content of Proposition 8.13 of [BF95].

As the reader will learn from section 7, in applications one frequently assumes that the underlying group $\pi_1(X)$ is freely indecomposable, and then the situation simplifies considerably as there can be no thin components in resolving band complexes (of course, assuming the arc stabilizers of T are trivial). Similarly, when one is concerned with hyperbolic groups, there can be no toral components.

One can also make a study of quasi-isometry types of leaves for a thin type lamination. Generic leaves are quasi-isometric to 1-ended trees, and in addition there are uncountably many leaves quasi-isometric to 2-ended trees. For details see Proposition 8.13 of [BF95] and, independently, Gaboriau [Gab96]. Of course, this is to be contrasted with the surface and toral types where the leaves are quasi-isometric to Euclidean space (of dimension 1 and > 1 respectively).

5.4 Remarks on the proof of the classification theorem

We now briefly describe the proof of Theorem 5.1. For details, see section 7 of [BF95]. As mentioned in the introduction, there is an alternative approach developed by Gaboriau, Levitt, and Paulin.

There is an algorithm for transforming a given band complex (say with a minimal lamination) to another one. When there is a free arc, we collapse from a maximal free subarc as described above. This is Process I. When there are no free subarcs, one performs a *sliding* move, called Process II. These are to be repeated producing a sequence of band complexes. There is a notion of complexity (non-negative half-integer valued). The moves never increase complexity, and whenever Process II is followed by Process I (i.e. whenever free subarcs disappear after a collapse) the complexity strictly decreases. It follows that eventually only Process I or only Process II is performed. In the first case the band complex is of thin type, and in the second one argues that it is of surface or of toral type.

6 Stable actions on $\mathbb{R}$-trees

Here is a sample statement that illustrates how the Rips machine can be applied to understand the structure of a finitely presented group that is acting on an $\mathbb{R}$-tree.

Theorem 6.1. *Suppose that a torsion-free finitely presented group G is acting non-trivially on an $\mathbb{R}$-tree T by isometries and that all arc stabilizers are trivial. Then one of the following holds.*

- *G splits as a non-trivial free product. In this case one can study the free factors by examining the induced action on T. Either this theorem can be applied to a given factor or this factor is a point stabilizer in T.*
- *G is a free abelian group.*
- *G is the fundamental group of a 2-complex X that contains as a subcomplex a compact connected surface S of negative Euler characteristic and $S \cap \overline{X \setminus S}$ is contained in ∂S. The inclusion induced homomorphism $\pi_1(S) \to \pi_1(X) = G$ is injective and each boundary component of S corresponds to an elliptic isometry in T. There is a filling geodesic measured lamination with measure of full support on S disjoint from ∂S*

("filling" means that each complementary component is either simply-connected or it contains a boundary component and its fundamental group is $\mathbb{Z}$).

Proof. We may assume that the action is minimal by passing to the minimal subtree, as in Proposition 2.9. Let K be a finite complex with $\pi_1(K) = G$ and choose a resolution $f : \tilde{K} \to T$ as in section 4.3. Convert K to a band complex X in the same manner as in Example 4.13. Then apply Theorem 5.1 to replace X by a band complex X' in "normal form".

If some component X'_i is of thin type, then X'_i can be assumed to contain a naked band and hence $G = \pi_1(X')$ splits as a non-trivial free product, so the first possibility holds.

If some component X'_i is of toral type, then either we obtain a free product decomposition of G using one of the points in $Fr(X'_i)$ (so the first possibility holds), or $G = \pi_1(X'_i)$ is free abelian, so the second possibility holds.

If some component X'_i is of surface type, then the third possibility holds.

Finally, if the lamination on X' is simplicial, then G acts on the simplicial tree dual to this lamination. The action can only be "freer" than the original action, so from Bass-Serre theory we conclude that the first possibility holds. □

E. Rips presented a proof of the following theorem at the conference at the Isle of Thorns in the summer of 1991. It answers affirmatively the conjecture of Morgan and Shalen.

Theorem 6.2 (Rips). *If G is a finitely presented group that acts freely by isometries on an $\mathbb{R}$-tree, then G is the free product of free abelian groups and closed surface groups.*

Proof. Decompose G into the free product of a free group and freely indecomposable factors, and apply Theorem 6.1 to each freely indecomposable factor. □

As indicated earlier, the methods generalize to stable actions. The following is stated as Theorem 9.5 in [BF95].

Theorem 6.3. *Let G be a finitely presented group with a stable action on an $\mathbb{R}$-tree T. Then either*

- *G splits over an extension E-by-cyclic where E fixes an arc of T, or*

- *T is a line and G splits over an extension of the kernel of the action by a free abelian group.*

The structure of the group G obtained in theorems above very much depends on the choice of the resolution. Imagine taking a sequence of finer and finer resolutions of the given stable action, as in the discussion preceding Proposition 4.11. Each band complex in the sequence gives rise to a splitting of G (more precisely, to a graph of groups decomposition of G).

Question. Does the sequence of splittings of G "stabilize"? In other words, is there a structure theorem for G that does not depend on the choice of a resolution, but only on the tree?

Question. Does Theorem 6.3 hold if "finitely presented" is replaced by "finitely generated" in the hypotheses?

Theorem 6.2 holds in the setting of finitely generated groups. Zlil Sela answered the above two questions affirmatively in the case that the action has the additional property that the stabilizers of tripods are trivial. This important case often arises in applications.

The structure of the group acting on an $\mathbb{R}$-tree without the assumption of stability is still very much a mystery.

Question. If a finitely presented group G admits a non-trivial isometric action on an $\mathbb{R}$-tree, does it also admit a non-trivial action on a simplicial tree (i.e. does it admit a non-trivial splitting)?

The answer is affirmative if G is a 3-manifold group by the work of Morgan and Shalen (Proposition 2.1 of [MS88b]).

7 Applications

I will now outline a number of applications of the theory of $\mathbb{R}$-trees. The technique can naturally be used in proofs of finiteness and compactness theorems. Surprisingly, as shown by the work of Zlil Sela, $\mathbb{R}$-trees can also be used to derive various structure theorems in group theory. It is impossible to cover all applications to date, so it seems reasonable to restrict this exposition to outlines of the most typical and the most striking applications. There is no discussion of the work of Rips and Sela on JSJ decompositions of finitely presented groups (see [RS97]), in part because in the meantime simpler proofs of more general theorems have been found [DS],[FP]. There is no doubt, however, that the intuition coming from the theory of $\mathbb{R}$-trees played a key role in this discovery.

7.1 Compactifying spaces of geometric structures

Topologists became interested in $\mathbb{R}$-trees with the work of Morgan and Shalen [MS84] that shed new light and generalized parts of Thurston's Geometrization Theorem. If M is a closed oriented n-manifold, then having a hyperbolic structure on M is equivalent to having a discrete and faithful representation $\pi_1(M) \to Isom_+\mathbb{H}^n$ into the orientation-preserving isometry group of the hyperbolic n-space, up to conjugation in $Isom_+\mathbb{H}^n$. For $n = 2$ and M of genus $g \geq 2$ the space

$$Hom_{DF}(\pi_1(M), Isom_+\mathbb{H}^2)/conj$$

of hyperbolic structures on M is the *Teichmüller space* of M. It is known that this space is homeomorphic to Euclidean space of dimension $6g-6$. The automorphism group $Aut(M)$ of M (homeomorphisms of M modulo isotopy, also known as the *mapping class group* of M) naturally acts on it (with finite isotropy groups), so the Teichmüller space is useful in the study of $Aut(M)$ as it plays the role of the classifying space.

An important ingredient of Thurston's theory of surface automorphisms [Thu88],[FLP79] is his construction of an equivariant compactification of the Teichmüller space. An ideal point is represented by a transversely measured geodesic lamination on M (measures that differ by a multiple are equivalent).

From the point of view of $\mathbb{R}$-trees, the construction of this compactification comes from the Compactness Theorem (see section 3.5). An ideal point is represented by a non-trivial and minimal isometric action of $\pi_1(M)$ on an $\mathbb{R}$-tree, with homothetic actions considered equivalent. Further, from Proposition 3.10 we see that the arc stabilizers are cyclic. Recall that an action of $\pi_1(M)$ on an $\mathbb{R}$-tree is small if it is minimal, does not have global fixed points, and all arc stabilizers are cyclic.

That the two approaches are equivalent follows from the following result of Skora [Sko96].

Theorem 7.1 (Skora [Sko96]). *If M is a closed hyperbolic surface, then any small action of $\pi_1(M)$ on an $\mathbb{R}$-tree is dual to a unique measured geodesic lamination on M.*

Proof. The proof using the Rips machine is considerably simpler than the original proof. We focus on the special case when the action is free; the general case is similar. Let $f : \tilde{X} \to T$ be a resolution of the action. Since $\pi_1(M)$ does not contain $\mathbb{Z} \times \mathbb{Z}$, X cannot have toral components, and since it is not freely decomposable, X cannot have simplicial or thin components, and in fact X must have a single surface component (see Theorem 6.1). Thus X can be taken to be a closed surface equipped with a measured geodesic lamination that fills the surface. To finish the proof, we need to argue that f is an exact resolution. If not, f factors as $f = gh$ through another resolution $g : \tilde{X}' \to T$, where $\tilde{h} : \tilde{X} \to \tilde{X}'$ is equivariant and sends leaves to leaves. By the same argument as above, X' can be taken to be a closed surface with a filling measured geodesic lamination. Thus $h : X \to X'$ is a homotopy equivalence that sends leaves to leaves and locally preserves the transverse measure. In the universal cover (identified with the hyperbolic plane), distinct leaves diverge from each other (in at least one direction), and therefore $\tilde{h}$ cannot send distinct leaves to the same leaf. Since $\tilde{h}$ induces a homeomorphism between the circles at infinity and a lamination is determined by the pairs of endpoints at infinity of its leaves, it follows that h can be taken to be a homeomorphism, showing that f is an exact resolution. □

Theorem 7.1 plays a prominent role in J.-P. Otal's proof [Ota96] of Thurston's Double Limit Theorem, which in turn is a key ingredient in the proof of the Hyperbolization Theorem for 3-manifolds that fiber over the circle.

In dimensions $n > 2$ the celebrated Rigidity Theorem of Mostow states that the space of hyperbolic structures on a closed manifold M^n has at most one point, and the construction using the Compactness Theorem is not particularly exciting in that case. However, it is important in Thurston's proof of the Geometrization Theorem to study the space

$$Hom_{DF}(G, Isom_+\mathbb{H}^n)/conj$$

where G is the fundamental group of a compact 3-manifold (with boundary) and $n = 3$. In particular, Thurston needed the fact that this space is compact when the 3-manifold is irreducible, aspherical, acylindrical, and atoroidal. In group-theoretic terms, this means that G is torsion-free and does not split over $1, \mathbb{Z}$, or $\mathbb{Z}^2$.

Theorem 7.2. *[BF95] Suppose G is finitely presented, not virtually abelian,* *and does not split over a virtually abelian subgroup. Then the space*

$$Hom_{DF}(G, Isom_+\mathbb{H}^n)/conj$$

of homotopy hyperbolic structures on G is compact.

Proof. If the space is not compact, there is a sequence going to infinity. The Compactness Theorem provides a small action of G on an $\mathbb{R}$-tree. Theorem 6.3 then implies that G splits over a virtually abelian subgroup. (Recall that a discrete group of isometries of $\mathbb{H}^n$ is either virtually abelian or it contains F_2.) □

This theorem generalizes earlier work of Thurston, Morgan-Shalen, and Morgan.

7.2 Automorphism groups of word-hyperbolic groups

It is the fundamental observation of F. Paulin [Pau91] that $\mathbb{R}$-trees arise also in the coarse setting of word-hyperbolic groups in the presence of infinitely many automorphisms of the group. The second part of the proof of the following theorem follows from the Rips machine.

Theorem 7.3. *Suppose G is a word-hyperbolic group such that $Out(G)$ is infinite. Then G splits over a virtually cyclic subgroup.*

Proof. Let $f_j : G \to G$ be an infinite sequence of pairwise non-conjugate automorphisms. Each f_j produces an isometric action ρ_j of G on its Cayley graph by sending $g \in G$ to the left translation by $f_j(g)$. The Compactness Theorem provides an action of G on an $\mathbb{R}$-tree T. The arc stabilizers of this action are small by Proposition 3.10, so the claim follows from Theorem 6.3. □

We will now assume that G is a torsion-free word-hyperbolic group. It is an open question whether every word-hyperbolic group has a torsion-free subgroup of finite index (or even whether it is residually finite). It is known that there are only finitely many conjugacy classes of finite order elements [Gro87].

For torsion-free G, an almost-converse of Theorem 7.3 holds [MNS]. If G splits as a free product $G = A * B$ with A and B nontrivial (infinite!), and if one, say A, is nonabelian, then for a fixed nontrivial $a \in A$ the automorphism

$f : G \to G$ that restricts to identity on B and to conjugation by a on A represents an element of infinite order in $Out(G)$. The remaining case is $G = \mathbb{Z} * \mathbb{Z} = F_2$, but $Out(F_2) = GL_2(\mathbb{Z})$ has many elements of infinite order. If G splits over $\mathbb{Z}$, say as $G = A *_C B$, with $A \neq C \neq B$ and $C =< c >$ infinite cyclic, then there is a *Dehn twist* automorphism $f : G \to G$ that restricts to identity on B and to conjugation by c on A. This represents an element of infinite order in $Out(G)$ as long as A and B are nonabelian. The case not covered by the "almost-converse" is when one of A or B is infinite cyclic. Finally, if G splits as $G = A*_C$ with $C =< c >$ infinite cyclic, then the automorphism (*Dehn twist*) that restricts to the identity on A and sends the "stable letter" t to tc has infinite order in $Out(G)$. For a more detailed discussion that includes cases with torsion see [MNS].

The above paragraph suggests the following sharpening of Theorem 7.3, at least for torsion-free, freely indecomposable word-hyperbolic groups. Call the subgroup of $Aut(G)$ generated by all inner automorphisms and all Dehn twists (with respect to all possible splittings over infinite cyclic subgroups) the *Internal Automorphism Group*, denoted $Int(G)$. It is a normal subgroup of $Aut(G)$. Note that the celebrated theorem of Dehn [Deh38] (see also [Lic64]) that the mapping class group of a closed orientable surface is generated by Dehn twists can be interpreted as saying $Int(G) = Aut(G)$ where G is the fundamental group of the surface. If the surface is allowed to be non-orientable and to have boundary, then the subgroup of the automorphism group (i.e. the homeomorphism group modulo isotopy) generated by Dehn twists has finite index.

Theorem 7.4 (Rips-Sela [RS94]). *If G is a torsion-free, freely indecomposable word-hyperbolic group, then the Internal Automorphism Group has finite index in $Aut(G)$.*

The proof introduces a new idea, the *shortening argument.*

Proof. Fix a finite generating set $\{\gamma_1, \cdots, \gamma_k\}$ for G which is closed under taking inverses and for $f \in Aut(G)$ define

$$d(f) = \max_{1 \leq i \leq k} ||f(\gamma_i)||$$

where $||\cdot||$ denotes the word length. In each coset of $Int(G)$ in $Aut(G)$ choose an automorphism f with minimal $d(f)$. Assuming that there are infinitely many cosets, we have an infinite sequence of automorphisms $f_1, f_2, \cdots \in$

$Aut(G)$ that represent distinct cosets of $Int(G)$ and each minimizes the function d in its coset. As in the proof of Theorem 7.3, we view each f_j as giving an action ρ_j of G on its Cayley graph. Note that $1 \in G$ is centrally located for ρ_j (or else composing f_j with an inner automorphism would produce a representative of the same coset with smaller d).

After passing to a subsequence, we obtain a limiting action ρ of G on an $\mathbb{R}$-tree T. We will examine this action and argue that for large j the automorphism f_j can be composed with a Dehn twist in such a way that d is reduced.

Let X be a finite band complex and $\phi : \tilde{X} \to T$ a resolution of ρ. Choose a basepoint $* \in \tilde{X}$ that maps to the basepoint in T. We can arrange (see Proposition 4.11) that the distance in $\tilde{X}$ between $*$ and $\gamma_i(*)$ equals the corresponding distance in T.

Since G is word-hyperbolic and so does not contain $\mathbb{Z} \oplus \mathbb{Z}$, X cannot have any toral components. Similarly, X cannot have any thin components, as we are assuming that G is freely indecomposable. Therefore, all components of X are of simplicial or surface type, and the simplicial pieces have infinite cyclic edge stabilizers.

Let us consider the two extreme cases. First suppose that X is a closed surface with a filling geodesic-like measured lamination. It is a fact of surface theory that there is a homeomorphism $h : X \to X$ fixing the basepoint, which can be taken to be a product of Dehn twists, such that the measure of each $h([\gamma_i])$ is arbitrarily small. This fact can be proved by "unzipping" the band complex (i.e. the "train-track", see [FLP79]) until the bands are arbitrarily thin and taking for h a homeomorphism that sends thick bands to thin bands. Now $f_j\pi_1(h)$ is a "shorter" representative of the coset $f_jInt(G)$ for sufficiently large j, a contradiction.

Now suppose that X is simplicial. Let T' be the simplicial tree dual to $\tilde{X}$. For notational simplicity we assume that T'/G is a single edge, corresponding to an amalgamated product decomposition of G (over $\mathbb{Z}$). The basepoint in $\tilde{X}$ corresponds to a vertex v in T'. Consider an edge e of T' that has v as an endpoint. The stabilizer of e is infinite cyclic. Say $c \in G$ generates this stabilizer. Let A denote the stabilizer of v and B the stabilizer of the other endpoint of e, so that $G = A *_{<c>} B$. Also, without loss of generality we can assume that the length of e is 1. The distance between v and $\gamma_i(v)$ in T' (equivalently, in T) is the minimal m_i such that γ_i is the product of the form $a_0 b_1 a_2 \cdots$ of total length $m_i + 1$ with the a's in A and the b's in B.

Fix a large j and consider the translates of the basepoint $1 \in G$ under the generators γ_i with respect to the representation ρ_j. After rescaling by the constant $d_j = d(f_j)$, the word-metric on G restricted to this finite set is close to the metric induced from T (or T') by restricting to the translates of v by the generators.

Let $b \in B$ be one of the b's occurring in the above representations of the γ_i's. The axis of $f_j(c)$ in G and the geodesic joining 1 and $f_j(b)$ are within 10δ for a length of about d_j and the translation length of $f_j(c)$ is $<< d_j$. Replace c by c^{-1} if necessary so that $f_j(c)$ translates from $f_j(b)$ towards 1. Choose the (positive) power m so that $f_j(c)^m$ translates $f_j(b)$ about halfway towards 1.

We now claim that precomposing f_j by the m^{th} power h of the Dehn twist that fixes A and conjugates B by c has the effect of shortening the representative of the coset $f_j Int(G)$. Indeed, write $\gamma_i = a_0 b_1 a_2 \cdots$ so that $f_j h(\gamma_i) = f_j(a_0) f_j(b_1)^{f_j(c)^m} f_j(a_2) \cdots$. The distance between 1 and $f_j h(\gamma_i)$ can be estimated in the usual way:

$$\begin{aligned} d(1, f_j h(\gamma_i)) \leq \quad & d(1, f_j(a_0)) + d(f_j(a_0), f_j(a_0) f_j(b_1)^{f_j(c)^m}) + \\ & d(f_j(a_0) f_j(b_1)^{f_j(c)^m}, f_j(a_0) f_j(b_1)^{f_j(c)^m} f_j(a_2)) + \cdots = \\ & d(1, f_j(a_0)) + d(1, f_j(b_1)^{f_j(c)^m}) + d(1, f_j(a_2)) + \cdots \end{aligned}$$

The terms of the form $d(1, f_j(a))$ are small compared to d_j (the ratio goes to 0), and the terms $d(1, f_j(b_k)^{f_j(c)^m})$ are approximately $d_j/2$. Thus the distance $d(1, f_j h(\gamma_i))$ is estimated above by about $m_i d_j/2$, and this is much less than $d(1, f_j(\gamma_i))$ (which is about $m_i d_j$).

The general case (when T' has perhaps more than one orbit of edges, or when X has both surface and simplicial components) is dealt with in the same way; only notation is more involved. $\square$

A version of the theorem can be proved for torsion-free word-hyperbolic groups that are free products using the classical theory of automorphisms of free products [FR40],[FR41].

The same method has other applications. Recall that a group G is *co-Hopfian* if every injective endomorphism $G \to G$ is surjective. Nontrivial free products are never co-Hopfian. For our purposes, the group $\mathbb{Z}$ is not freely indecomposable (it splits over the trivial group).

Theorem 7.5 (Sela [Sel97]). *Every freely indecomposable word-hyperbolic group is co-Hopfian.*

Proof. Let $Inj(G)$ denote the semi-group of injective endomorphisms of G. The idea is to follow the above argument and show that $Aut(G)$ has finite index in $Inj(G)$. The only difference with the situation $Int(G) \subset Aut(G)$ is that $Inj(G)$ is not a group and $Aut(G)$ is not normal in $Inj(G)$, but those features were never used. Finally, note that if $Aut(G)$ has finite index in $Inj(G)$, then a nontrivial power of every $f \in Inj(G)$ is an automorphism, and so $Inj(G) = Aut(G)$. □

Recall that a group G is *Hopfian* if every surjective endomorphism $G \to G$ is an isomorphism. Z. Sela has announced the following result [Sela]:

Theorem 7.6. *Every torsion-free word-hyperbolic group is Hopfian.*

The proof uses more elaborate ideas and will not be outlined here.

Theorem 7.7 (Gromov [Gro87], Sela [Sel97]). *Let Γ be a finitely presented torsion-free freely indecomposable group and let G be a word-hyperbolic group. Then there are only finitely many conjugacy classes of subgroups of G isomorphic to Γ.*

Proof. First consider the simple case when Γ does not admit any splittings over $\mathbb{Z}$. Then we argue that there can be only finitely many conjugacy classes of monomorphisms $f : \Gamma \to G$. For suppose that there are infinitely many. Let $f_1, f_2, \cdots : \Gamma \to G$ be an infinite sequence of pairwise non-conjugate monomorphisms. We thus get a sequence of actions ρ_i of Γ on G: $\rho_i(\gamma)$ acts by left translation by $f_i(\gamma)$. By conjugating each f_i we may assume that $1 \in G$ is centrally located with respect to each ρ_i (and with respect to a fixed finite generating set for Γ). Now pass to a subsequence and obtain an action of Γ on an $\mathbb{R}$-tree. As before, this action induces a splitting of Γ over $\mathbb{Z}$.

If Γ admits a splitting over $\mathbb{Z}$, then we could precompose a given monomorphism $\Gamma \to G$ by automorphisms of Γ and obtain an infinite sequence of non-conjugate monomorphisms $\Gamma \to G$. This phenomenon is precisely what the shortening argument is designed to handle. Given a monomorphism $\Gamma \to G$, conjugate it by an element of G and precompose by an automorphism of Γ so as to make $1 \in G$ centrally located and to make the maximal displacement of 1 smallest possible. Now the claim is that there can be only finitely many such minimizing monomorphisms. The proof of the claim is analogous to the proof of Theorem 7.4. If there are infinitely many such monomorphisms, consider the limiting tree and use it to construct an automorphism $h : \Gamma \to \Gamma$ that can be used to shorten representations ρ_j for large j. □

7.3 Fixed subgroup of a free group automorphism

Let F_n be the free group of rank n and $f : F_n \to F_n$ an automorphism. Recall that F_n contains free subgroups of infinite rank. The following theorem was conjectured by Peter Scott.

Theorem 7.8 (Bestvina-Handel). *The rank of the subgroup $Fix(f)$ of elements of F_n fixed by f is at most n.*

The proof in [BH92] does not use the theory of $\mathbb{R}$-trees. Z. Sela [Sel96] and D. Gaboriau-G. Levitt-M. Lustig [GLL98] found a simpler argument using $\mathbb{R}$-trees. We now outline their ideas.

First, for $k = 1, 2, \cdots$ let g_k be an automorphism conjugate to f^k such that $1 \in F_n$ is centrally located with respect to the representation ρ_k that to $\gamma \in F_n$ associates left translation $F_n \to F_n$ by $g_k(\gamma)$. This conjugation is necessary in order to apply the Compactness Theorem, but of course $Fix(g_k)$ is in general different from $Fix(f)$. It is therefore more natural to consider elements of F_n fixed up to conjugacy. If f has finite order as an outer automorphism, the rescaling constants remain bounded. Such automorphisms were handled by Culler [Cul84] who showed that the fixed subgroup is either cyclic or a free factor. For non-periodic automorphisms, we analyze the action of F_n on the $\mathbb{R}$-tree T obtained as the limit of a subsequence of representations ρ_k above.

The key observation is that any $\gamma \in F_n$ which is fixed up to conjugacy by f is elliptic in T. Indeed, the translation length of γ can be computed as the limit of ratios

$$\frac{\text{translation length of } g_k(\gamma)}{\text{rescaling factor for } \rho_k}$$

and this converges to 0 since the denominators go to infinity, while the numerators are constant (and equal to the length of the conjugacy class of γ). The same argument shows that periodic conjugacy classes are elliptic in T (and also those that grow slower than the fastest growing conjugacy classes).

Second, we construct a bilipschitz homeomorphism $H : T \to T$ which is equivariant with respect to f, i.e. $h(\gamma(x)) = f(\gamma)(h(x))$. This construction is due to Sela who used it extensively. He calls it the "basic commutative diagram". First form the group $G = F_n \rtimes_f \mathbb{Z} =< F_n, t | tgt^{-1} = f(g) >$, the mapping torus of f. Each action ρ_k extends to an action $\tilde{\rho}_k$ of G on F_n by sending t to the conjugate of f by the same element used to conjugate f^k. Of course, the extended action is not isometric, only bilipschitz. Pass to a subsequence as usual to obtain a bilipschitz action of G on an $\mathbb{R}$-tree. Restricting to F_n gives the discussion of the first paragraph, while $t \in G$ provides the desired bilipschitz homeomorphism $H : T \to T$.

Third, we promote H to a homothety. This is not absolutely necessary here, but in other applications it comes handy. The following construction is due to Paulin [Pau97a]. The Compactness Theorem implies that the space $\mathcal{PED}_0$ of projectivized nontrivial 0-hyperbolic equivariant distance functions on F_n is compact. The preimage of the closed subset $\mathcal{PED}_0^T$ of $\mathcal{PED}_0$ consisting of those projective classes of distance functions d with the property that

$$(x \cdot y)_T \geq (x \cdot z)_T \Rightarrow (x \cdot y)_d \geq (x \cdot z)_d$$

in $\mathcal{ED}$ is a convex cone: If d_1 and d_2 are two 0-hyperbolic distance functions satisfying the above condition, then $sd_1 + (1-s)d_2$ is also such a distance function for $0 \leq s \leq 1$. Subscripts T and d above indicate the metric with respect to which $(\cdot, \cdot)$ is taken. It easily follows that $\mathcal{PED}_0^T$ is a compact absolute retract, and therefore has the fixed point property. By pulling back, H induces a homeomorphism of $\mathcal{PED}_0^T$. A fixed point of H determines a new 0-hyperbolic distance function on F_n with respect to which H is a homothety. By Connecting the Dots (Lemma 2.13), we obtain a new tree that we continue to denote by T. We remark that the new tree may not be homeomorphic to the old, but is rather obtained from the old by collapsing some subtrees. What is important is that arc stabilizers in the new tree are contained in the arc stabilizers of the old tree.

Alternatively, steps 1-3 could have been avoided by quoting some of the theory developed in [BH92]. See [Lus], where this alternative construction is carried out in detail.

We now arrive at the heart of the argument.

Proposition 7.9. *Assume that F_n acts on an $\mathbb{R}$-tree T and the action is small. Then all vertex stabilizers of T have rank $\leq n$. Further, if there is a vertex stabilizer V of rank n, then the action is simplicial, all edge stabilizers are infinite cyclic, and every vertex stabilizer that is not infinite cyclic is conjugate to V.*

Before giving the proof of Proposition 7.9 we finish the proof of Theorem 7.8. We have seen above that each $\gamma \in Fix(f)$ is elliptic in T. It is an exercise to show that there is a point $v \in T$ fixed by each $\gamma \in Fix(f)$. The ingredients are 1) the product of two elliptic isometries of T with disjoint fixed point sets is hyperbolic, and 2) arc stabilizers of T are cyclic. We may assume that $rank(Fix(f)) > 1$, and then v is unique. Since f leaves $Fix(f)$ invariant, equivariance forces H to fix v. In particular, H induces an automorphism $f_v : Stab(v) \to Stab(v)$. If $rank(Stab(v)) < n$, we can apply induction on the rank and conclude that $rank(Fix(f)) = rank(Fix(f_v)) < n$. If $rank(Stab(v)) = n$ and f_v has finite order (as an outer automorphism) we can apply Culler's result to conclude that $rank(Fix(f)) \leq n$. If $rank(Stab(v)) = n$ and f_v has infinite order, we can repeat the construction with f_v in place of f. We obtain a sequence of automorphisms $f = f_0, f_v = f_1, f_2, \cdots$. We can stop when the rank of the vertex stabilizer is $< n$ or when the restriction of the automorphism to the vertex stabilizer has finite order. It remains to argue that the sequence must terminate. The tree T constructed above provides a graph of groups decomposition $\mathcal{G}_0$ of F_n with cyclic edge groups (according to Proposition 7.9). The only vertex group is $Stab(v)$ and it has rank n. The next iteration provides a graph of groups decomposition $\mathcal{G}_1$ of $Stab(v)$ of the same nature. We claim that $\mathcal{G}_1$ can be used to refine $\mathcal{G}_0$, by "blowing up" the vertex. Indeed, this will be possible if all edge groups of $\mathcal{G}_0$ are elliptic in $\mathcal{G}_1$. But the edge groups of $\mathcal{G}_0$ are permuted (up to conjugacy) by f, since the orbits of edges are permuted by H, and the claim follows from the observation above that f-periodic conjugacy classes are elliptic in T. If the sequence of automorphisms does not terminate, then continuing in this fashion we obtain graph of groups decompositions of F_n with all edge groups cyclic, with only one vertex, and with more and more edges. This is not possible, for example by the generalized accessibility theorem of [BF91], or better yet, by abelianizing there can be at most n edges.

Proof of Proposition 7.9. Inductively, we assume that Proposition 7.9 holds for free groups of rank $< n$. If $Stab(v)$ is contained in a proper free factor of F_n, then the statement follows inductively on the rank of the underlying free group.

Claim 1. If T is simplicial then all vertex stabilizers have rank $\leq n$. If there is a vertex stabilizer of rank n, then all other vertex stabilizers have rank 1 and all edge stabilizers are infinite cyclic.

An edge of T with trivial stabilizer induces a free factorization of F_n which implies the claim by induction. So we can assume that all edge stabilizers are infinite cyclic. Now construct the graph of spaces associated with the graph of groups T/F_n as in [SW79]. Every vertex in T/F_n is represented by a rose, and every edge by an annulus. Since adding annuli does not change the Euler characteristic, we see that the Euler characteristic of the resulting space, which must be $1-n$, is equal to the sum $\sum(1-r_i)$, where $r_1, r_2, \cdots$ denote the ranks of the vertex labels in T/F_n. Since $r_i > 0$ for each i by assumption, Claim 1 follows.

Assume now $rank(Stab(v)) < \infty$ and let $\tilde{X} \to T$ be a resolution of T such that a compact set K in some complementary component $D \subset X$ satisfies $im[\pi_1(K) \to \pi_1(X)] = Stab(v)$ (see Proposition 4.11). Since F_n does not contain $\mathbb{Z} \times \mathbb{Z}$ nor an extension of $\mathbb{Z} \times \mathbb{Z}$ by $\mathbb{Z}$, X cannot have any toral components. Likewise, if X has a thin component, we can transform it so that there is a naked band disjoint from K and we conclude that $Stab(v)$ is contained in a proper free factor. Therefore, X consists of surface and simplicial components. If there is at least one surface component (of negative Euler characteristic), then the Euler characteristic count of Claim 1 implies $rank(Stab(v)) < n$. So assume that all components of X are simplicial, and let T' be the dual (simplicial) tree. If an edge stabilizer in T' is trivial, then all vertex stabilizers in T', including $Stab(v)$, have rank $< n$. The following claim concludes the proof in case $rank(Stab(v)) < \infty$:

Claim 2. If all edge stabilizers in T' are infinite cyclic, then T is simplicial.

To prove the claim, for each primitive $a \in F_n$ consider the subtree $T'_a \subset T'$ consisting of points fixed by a power of a. First note that these subtrees are finite. Indeed, if e is an edge in T'_a whose stabilizer is $< a^k >$, then exactly k edges in T'_a (namely, the translates of e by a) can project to the same edge in the quotient. By $T_a \subset T$ denote the image of T'_a under the natural map $\pi' : T' \to T$ (see section 4.4). Then T_a is a finite tree (by "local injectivity" – see section 4.4). Moreover, by the equivariance of π, T_a is fixed pointwise by a power of a. It then follows that if a and b are primitive elements with $a \neq b^{\pm 1}$, then T_a and T_b can intersect in at most a point. Since there are only finitely many orbits of the T_a's, the claim follows.

It remains to rule out the possibility that $rank(Stab(v)) = \infty$. Choose a free factor H of $Stab(v)$ with $n < rank(H) < \infty$. Let $\tilde{X} \to T$ be a resolution such that H is in the image of $\pi_1(D) \to \pi_1(X)$ for a complementary component D that corresponds to the orbit of v. Then $im[\pi_1(D) \to \pi_1(X)]$ is contained in $Stab(v)$ and contains H, so that its rank is $> n$. Now analyze the components of X in a similar way as above to reach a contradiction. $\square$

For more details see Gaboriau-Levitt [GL95]. They also bound the number of orbits of branch points and their "valences" for small actions of F_n.

7.4 The topology of the boundary of a word-hyperbolic group

Let G be a word-hyperbolic group and ∂G its boundary. The following theorem was the motivating goal of [BM91].

Theorem 7.10. *If G has one end, then ∂G is connected and locally connected.*

The first part of the conclusion (that ∂G is connected) was proved in [BM91], but the second was proved only under the assumption that ∂G contains no cut points. The theory of $\mathbb{R}$-trees was used to establish:

Theorem 7.11 (Bowditch, Swarup). *If G has one end, then ∂G contains no cut points.*

Sketch of proof. For every compact metric space M, Bowditch [Bowb] constructs a canonical map $M \to D$ to a dendrite D. A compact metric space is a *dendrite* if it is locally connected and each pair of points x, y is joined by a unique arc, denoted $[x, y]$. This is done as follows. Two points $x, y \in M$ are NOT equivalent if there is a collection C of cut points in M that each separate x from y and which is order-isomorphic to the rationals. Bowditch argues that the quotient space D is a dendrite. Apply this construction to $M = \partial G$. Since G acts on ∂G, there is an induced action of G on D. If ∂G has a cut point, then it contains a lot of cut points (translates of the original), and Bowditch argues that D is not a point. Further, he shows that the action of G on $T = D \setminus \{endpoints\}$ has trivial arc stabilizers and is nonnesting, in the sense that if J is an arc in T and $g(J) \subseteq J$, then $g(J) = J$ (and hence $g = 1$). The tree T is homeomorphic to an $\mathbb{R}$-tree, but there is no reason why there should be an equivariant $\mathbb{R}$-tree metric on T. If there were, we could apply Theorem 6.3 and conclude that G splits over a 2-ended group. This is where Sacksteder's theorem comes in. We can construct a resolution $\tilde{X} \to T$ as before, but the lamination on X will not have a transverse measure. Theorem 4.8 provides a transverse measure (perhaps not of full support). It is easy to see that the arc stabilizers of the dual $\mathbb{R}$-tree are trivial. Thus G splits over a 2-ended group.

The proof was completed by Swarup [Swa96]. The idea is to keep refining the splitting as in the proof of Theorem 7.8. So suppose inductively that $\mathcal{G}$ is a graph of groups decomposition of G with 2-ended edge groups. If E is an edge group, the endpoints of the axis of an element of E are identified in the dendrite D [Bowa]. Each vertex group is word-hyperbolic and it is quasi-convex in G. Let $\Lambda(V)$ denote the limit set of a vertex group V of $\mathcal{G}$. It can be argued [Bowa] that for at least one vertex group V the image of $\Lambda(V)$ in D is not a single point. It follows that the induced action of V on T is nontrivial, has trivial arc stabilizers, and all edge groups contained in V are elliptic. Now apply Sacksteder's theorem again to replace T by an $\mathbb{R}$-tree T' on which V acts nontrivially by isometries and with trivial arc stabilizers. The important point is that it can be arranged that the edge groups in V remain elliptic in T'. We then obtain a nontrivial splitting of V over two-ended groups that can be used to refine the graph of groups decomposition $\mathcal{G}$.

The final contradiction comes from the generalized accessibility theorem of [BF91] that provides an upper bound to the number of edges of a reduced graph of groups decomposition of a finitely presented group with small edge groups. A graph of groups is *reduced* if for each valence one and two vertex of the graph, the label of the vertex properly contains the labels of incident edges. This theorem, combined with the fact that in word-hyperbolic groups the ascending chain condition holds for 2-ended subgroups, implies that from some point on in the construction the refinement of the graph of groups consists of the introduction of new valence two vertices and edges whose labels are finite index subgroups in the old edge labels. Using a technique of Dunwoody [Dun85], one now shows that G splits over a finite subgroup, contradiction. Indeed, the sequence of graphs of groups gives rise to a sequence of pairwise disjoint tracks in a finite complex representing G. The tracks must eventually be parallel to previous tracks. Thus one of the tracks accounts for infinitely many edges in the splitting, i.e. the image of its fundamental group in G must be contained in infinitely many edge groups. But then this image is contained in the intersection, which is finite, and hence the splitting of G over a finite subgroup. $\square$

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Index