A class of pattern–forming models

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Abstract

A general class of nonlinear evolution equations is described, which support stable spatially oscillatory steady solutions. These equations are composed of an indefinite self-adjoint linear operator acting on the solution plus a nonlinear function, a typical example of the latter being a double-well potential. Thus a Lyapunov functional exists. The linear operator contains a parameter ρ which could be interpreted as a measure of the pattern-forming tendency for the equation. Examples in this class of equations are an integrodifferential equation studied by Goldstein, Muraki and Petrich and others in an activator-inhibitor context, and a class of fourth order parabolic PDE's appearing in the literature in various physical connections and investigated rigorously by Coleman, Leizarowitz, Marcus, Mizel, Peletier, Troy, Zaslavskii, and others. The former example reduces to the real Ginzburg-Landau equation when $\rho = 0$.

The most complete results, including threshold results for the appearance of globally minimizing patterns and many other properties of the patterns themselves, are given for complex-valued solutions in one space variable. A complete linear stability analysis for all such sinusoidal solutions is also given; it extends the set of stable solutions considerably beyond the global minimizers.

Other results, including threshold results and the existence of large amplitude patterns as well as of bifurcating solutions, are provided for real-valued solutions; these results are relatively independent of the number of space variables. Finally, a slightly different class of evolution equations is given for which no patterned global minimizer exists, but a sequence of patterned solutions exist whose instabilities (if they are unstable) become ever weaker and the fineness of the oscillation becomes ever more pronounced.

1 Introduction

Stable and metastable spatial patterns have been observed, modeled, and simulated in a great many contexts in the natural sciences. The following noteworthy works represent some of these contexts, other than the more traditional fluid dynamical ones: Turing [35] (biological differentiation), Cahn [3] (spinodal decomposition of alloys), Swift and Hohenberg [34] (thermal convection with fluctuations; condensation of liquid), Meinhardt [24] (biological patterns), Ohta–Kawasaki [30] (diblock copolymers), Meron [25] (general excitable systems), Seul–Andelman [33] (contains references to lipids, magnetic films, ferrofluids, amphiphilic layers at an interface, polymeric systems), and Goldstein–Muraki–Petrich [16] (activator-inhibitor systems, superconductors, magnetic fluids). This list is far from complete.

From a mathematical point of view, models in these papers have most commonly taken the form of evolution equations of relaxational type (gradient flow for a free energy functional) and/or reaction-diffusion type.

In this paper we present, and analyze rigorously, a framework for understanding a large class of relaxational models forming stable patterns, which is also relevant to some models of activator-inhibitor type. In later sections we pay special attention to two prototypes belonging to our class:

P1. A second order parabolic PDE with additional interaction modeled by an integral operator. Such models have appeared in [30, 33, 16]. Models bearing some similarity were studied in [29, 19, 2]. This prototype is examined in Sec. 5.

P2. A class of fourth order parabolic PDE's of Swift-Hohenberg type. This type of equation was discussed in [34, 17, 16, 33] and other places. It was also proposed by Coleman and Mizel (reference in [18]) as a model for materials with alternating phases, and its generalizations were studied intensively in [18, 7, 26]. See Sec. 6.

Our models have an explicit parameter ρ which could be interpreted as a generalized Bond number [33] or as a measure of the equation's pattern-forming strength. There is a threshold phenomenon, in the form of a critical value of ρ below which no globally minimizing patterns are possible, and beyond which they do exist.

The general setup, which is explained in Sec. 2, involves self-adjoint operators in L_2 spaces of periodic functions of a space variable. The evolution equations are gradient flows for energy functionals which consist of an indefinite part involving these operators plus a term involving a double or single well function growing more rapidly than quadratically. These energy functionals in some cases take the form of well-known Ginzburg-Landau functionals to which an aggregation effect has been added.

We deal with complex-valued functions in the first part of the paper; explicit sinusoidal solutions are available in that case, and stability and many other properties of these minimizing patterns can be derived. In the second part, however, we specialize to real-valued functions, realizing that they are the most relevant in many applications.

Stability in one sense can be settled by showing that a solution ϕ is a global minimizer of the associated energy. Namely, in that case there exists no solution $u(t)$ of the evolution problem leaving ϕ. Such a solution $u(t)$ would be defined for all $t < 0$, $u(-\infty) = \phi$, and $u(t) \neq \phi$ for $t > -\infty$; it is ruled out because the energy would have to decrease on $u(t)$. In many cases, this globally minimizing property implies stability in the Lyapunov sense in the context of some Banach space (see Remark 1 of Thm. 1). This type of stability is the main focus of our results. However in the complex-valued case our minimizers are sinusoidal, and this permits a (weaker) linearized stability analysis also to be performed. We find in Sec. 8 that many more solutions satisfy this weaker stability statement.

The basic results in the complex-valued case are explained in Sec. 3, and proved in Sec. 4. An activator-inhibitor paradigm with infinite inhibitor kinetics (as discussed in [16]) is shown in Sec. 5 to fit into our theory. That is the above-mentioned prototype P1. In that section, one of the self-adjoint operators is an integral operator generalizing the "spreading" effect associated with the diffusion of the inhibitor.

The fourth order prototype P2 is treated in section 6. In section 7, we discuss a class of models for which the energy functional has a minimizing sequence along which the patterns become more and more finely oscillatory.

Sec. 8 is devoted to the linear stability analysis of complex sinusoidal solutions of our class of evolution equations. A complete characterization is given of those solutions which are stable in this sense.

As mentioned before, the second part of the paper extends much of the preceding theory to the case when our functions are restricted to be real-valued. We obtain estimates for the threshold value of the parameter ρ. The existence of real-valued nonconstant stationary solutions of the evolution equation with arbitrarily small wavelength, for ρ large enough, follows from our analysis. Such solutions in the complex-valued case are given explicitly already in Sec. 3.

During most of the paper, large amplitude stable patterns are the main focus of attention. However, stable patterns also arise bifurcating from a spatially uniform steady solution which loses its stability when ρ surpasses some bifurcation point. These bifurcating solutions have small amplitude. Their existence is dealt with in section 10, in the real-valued case. Thus typically both kinds of patterned solutions exist. A natural question is the following: when ρ increases so that stable patterns appear, do the first ones which appear have large or small amplitudes, i.e. do they or do they not bifurcate from uniform steady solutions? We show in section 11 that a sufficient condition exists, depending only on the nonlinearity, under which the first patterns are not the bifurcating ones. Moreover in the case of the popular quartic double-well defined type nonlinearity, this condition is fulfilled.

Further properties of real-valued minimizers are given in Secs. 12 and 13. The paper ends with comments and a summary in section 14.

2 The context

Let A and B be self-adjoint densely defined closed negative linear operators on the L_2 space of complex-valued λ-periodic functions of x. We use the scalar product

$$\langle u, v\rangle = \frac{1}{\lambda}\int_0^\lambda u\bar{v}dx.$$

The wavelength λ is a variable parameter. Generally, we think of it as being large; then the *minimal* period of our solutions will typically be much smaller than λ.

Our assumptions on A and B are:

A1. Smooth functions $\hat{A}(k)$ and $\hat{B}(k)$ defined for all real $k \geq 0$, independent of λ, exist such that when e^{ikx} is λ-periodic, $\hat{A}(k) = e^{-ikx}A[e^{ikx}]$, and the same for $\hat{B}$. These symbols are real and even.

A2.
$$\lim_{k\to\infty} \frac{\hat{A}(k)}{\hat{B}(k)} = \infty, \quad \limsup_{k\to\infty} \hat{B}(k) < 0. \tag{1}$$

A3. The nullspaces of A and B are the set of constant functions. It follows that $\hat{A}(k)$ and $\hat{B}(k)$ are strictly negative for $k > 0$.

We have this Fourier representation: if $u = \sum_m u_m e^{i2\pi mx/\lambda}$, then $Au = \sum_m u_m \hat{A}(2\pi m/\lambda) e^{i2\pi mx/\lambda}$, and the domain of A is the set of functions u such that this series converges in L^2. A similar statement holds for B. Note that **A2** (1) implies the inclusion of domains $\mathcal{D}(A) \subset \mathcal{D}(B)$.

Let $F(w)$ be a real differentiable function of $w \geq 0$, with a unique minimum of 0 at $w = u_0^2 \geq 0$, such that

$$\lim_{w\to\infty} F'(w) = \infty. \tag{2}$$

We note that $F(w) = (u_0^2 - w)^2$ $(u_0 > 0)$ and $F(w) = \alpha w + w^2$, $\alpha \geq 0$, satisfy these assumptions. In the former case, when u is real $F(u^2)$ is a double–well function with equal depth wells at $u = \pm u_0$; in the latter case $u_0 = 0$ and it is a single–well function. Let $f(u) = 2uF'(|u|^2)$.

We consider stable patterned solutions of the evolution equation

$$u_t = Au - \rho Bu - f(u), \tag{3}$$

which we assume has a global solution in $\mathcal{D}(A) \cap L_\infty$ for each $t \geq 0$, given any initial condition in that space.

This evolution equation is a gradient flow for the energy functional

$$E[u] = -\frac{1}{2}\langle Au, u\rangle + \frac{\rho}{2}\langle Bu, u\rangle + \frac{1}{\lambda}\int_0^\lambda F(|u|^2)dx. \tag{4}$$

Our main goal will be to investigate the possibility of *stable* stationary nonconstant solutions of (3). Our main emphasis is on global minimizers of E, because they provide the strongest stability statements. However, in the case of complex sinusoidal solutions, we also examine our solutions from the point of view of a linear stability criterion in Sec. 8, following the procedure of Newton and Keller [27, 28]. We shall call a minimizer u^ρ *nontrivial* if $u^\rho \neq$ const. In view of **A3** and $F \geq 0$, a sufficient condition for this is that $E[u^\rho] < 0$, since constants have $E \geq 0$.

I. Complex-valued minimizers

3 The minimizers and their properties

The main results in Part I have to do with criteria for sinusoidal functions u to be global minimizers of E. Later in Sec. 8 we consider solutions that are not necessarily minimizers, but which satisfy a linear stability criterion. As expected, we find that the class of solutions satisfying this criterion is much larger than that of global minimizers.

To formulate our results we need to define some functions and critical values. Let

$$M(\rho,k)=\hat{A}(k)-\rho\hat{B}(k) \tag{5}$$

(an increasing function of ρ), and let

$$\rho^*(\lambda)=\inf\{\rho : M(\rho,k_m)>2F'(u_0^2+) \text{ for some } m\}, \tag{6}$$

where

$$k_m=\frac{2\pi m}{\lambda}, \quad m=1,2,\ldots. \tag{7}$$

In view of **A1** and **A2**, we have $0<\rho^*(\lambda)<\infty$.

Note the following alternate characterization of ρ^* in the case $u_0>0$:

$$\rho^*(\lambda)=\inf_m \frac{\hat{A}(k_m)}{\hat{B}(k_m)}. \tag{8}$$

In fact, in this case $F'(u_0^2)=0$. To derive (8), observe that $M(\rho,k_m)=-\hat{B}(k_m)[\rho-\frac{\hat{A}(k_m)}{\hat{B}(k_m)}]$; and this is positive for some value of m exactly when ρ is greater than ρ^* as defined by (8).

Below, we speak of functions ae^{ikx}. The solution set of (3) and the set of minimizers of E are invariant under multiplication by a constant $e^{i\theta}$. Therefore we may, and shall, always assume that a is real and nonnegative. In this sense, our minimizers are really equivalence classes.

Part (e) below utilizes a function $\phi(t)$ defined as follows. Let $F_c(w),\ w>u_0^2$, be the greatest monotone increasing function with $F_c(w)\le F(w)$ for every w in that range. Its graph is the convex hull of that of F. We have that $F_c'(w)$ is nondecreasing, and by (2), $F_c'(\infty)=\infty$. Since $F_c'(w)$ takes on all values from $\gamma=F'(u_0^2)$ to ∞, there exists a function $\phi(t),\ t\ge\gamma$, such that $F_c'(\phi(t))=t$.

The following theorem provides existence and nonexistence results for nontrivial minimizers, as well as several properties enjoyed by them. See the explanatory comments after the statement of the theorem.

Theorem 1 *(a) For each $\lambda > 0,\ \rho > \rho^*(\lambda)$, there exists a nontrivial global minimizer of E of the form ae^{ikx} for some $a = a(\rho,\lambda) > u_0,\quad k = k(\rho,\lambda) > 0$. If $u_0 > 0$, this is also true for $\rho = \rho^*$; then $a(\rho^*(\lambda),\lambda) = u_0$.*

(b) For $\rho < \rho^*(\lambda)$, the only global minimizers are constants with $|u| = u_0$.

(c) The functions a and k satisfy, for some $0 < m_0 < \infty$,

$$\lim_{\rho\downarrow\rho^*(\lambda)} a(\rho,\lambda) = u_0, \quad \lim_{\rho\to\infty} a(\rho,\lambda) = \infty, \lim_{\rho\downarrow\rho^*(\lambda)} k(\rho,\lambda) = \frac{2\pi m_0}{\lambda}. \tag{9}$$

(d) If

$$\inf \hat{B}(k) < \hat{B}(k) \quad \textit{for each} \quad k \tag{10}$$

(the former could be $-\infty$), then $\lim_{\rho\to\infty} k(\rho,\lambda) = \infty$.

(e) Let $\phi(t)$ be the function defined above. Then for some C_i with $C_2 > 0$,

$$a^2(\rho,\lambda) \geq \phi(C_1 + C_2\rho) \tag{11}$$

when $C_1 + C_2\rho \geq \gamma$.

(f) Let

$$\rho_0^*\,(\geq 0) = \textit{ the critical value of } \rho \textit{ beyond which } M(\rho,k) > 0 \textit{ for some } k$$

(unconstrained by the requirement (7)). For $\rho > \rho_0^*$, let $\bar{k}(\rho) > 0$ be the least value of k at which $M(\rho,k)$ is maximized (neither ρ_0^* nor $\bar{k}$ depend on λ). For $\rho > \rho_0^*$,

$$0 < \bar{k}(\rho) = \liminf_{\lambda\to\infty} k(\rho,\lambda) \leq \limsup_{\lambda\to\infty} k(\rho,\lambda) < \infty \tag{12}$$

and

$$u_0 \leq \liminf_{\lambda\to\infty} a(\rho,\lambda) \leq \limsup_{\lambda\to\infty} a(\rho,\lambda) < \infty. \tag{13}$$

(g) If the function $\frac{\hat{A}(k)}{\hat{B}(k)} \geq 0$ attains a strict minimum at $k = 0$, then there exists a number Λ, depending only on A and B, such that the number $m_0 = 1$ in (9) when $\lambda > \Lambda$.

Remarks.

1. Items (a) and (b) are threshold results, showing that nontrivial global minimizers exist for $\rho > \rho^*(\lambda)$ $(\geq \rho^*(\lambda)$ in the case $u_0 > 0)$ and not for $\rho < \rho^*(\lambda)$. Moreover, these minimizers are given explicitly. In Lemma 2 below, it will be shown that in typical cases, these are the only global minimizers. Although nontrivial global minimizers do not exist for $\rho < \rho^*$, stable patterns generally do exist, if stability is interpreted according to a linearized criterion. This is brought out in Sec. 8.

In the case $\frac{d^2}{du^2}F(u^2) > 0$ at $u = a$, then it can be show that the minimizer under consideration is also stable in the sense of Lyapunov in the space L_2; the proof will not be given.

2. Items (c), (d), and (e) give information about the amplitude and minimal period of the global minimizers which were constructed in part (a). The properties listed are proved only for them, with no claim that they hold for all global minimizers; however it is clear that in typical cases they will. The amplitude a grows without bound as $\rho \to \infty$, and under a reasonable additional assumption (10) the wavenumber k does as well. Thus the minimal wavelength shrinks. The relation (11) gives a more precise lower bound for the rate of growth of the amplitude. For the case $F(w) = (w - u_0^2)^2$, we have $a > C\rho^{1/2}$ as $\rho \to \infty$. It should be noted that the order of growth of this lower bound depends only on the function F. It was shown in [26] that such a bound may sometimes be improved by taking into consideration the operators A and B as well. In fact in the case of the fourth order problem given in Sec. 6 below and this same convex function F, with the minimization problem restricted to real valued functions, those authors established that the amplitude grows at least at the rate $O(\rho)$ as $\rho \to \infty$.

As ρ approaches its threshold value ρ^* from above, the amplitude approaches the position u_0 of F's well, and the wavenumber approaches a finite value. Part (g) indicates, under the condition given there, that when ρ is near ρ^*, the wavelength of the pattern is equal to λ when the latter is large, i.e. the wavelength is the largest possible, given the periodicity constraint on u.

3. Item (f) explores the effect on the global minimizer when the size of the basic period interval becomes very large. We conclude that there is no important effect. The wavenumber k and amplitude a are bounded above and below independently of λ. An examination of the proof will show, in fact, that in typical cases they approach finite limits as $\lambda \to \infty$. This means that the structure of stationary patterns is little affected by the domain size λ, when the latter is large.

4 Proofs

Lemma 1 *Let $u_{m,a}(x) = ae^{ik_m x}$, $a > 0$ real, k_m given by (7). There exist functions $a^*(\rho, \lambda)$ and $\rho_1(\lambda) \le \rho^*(\lambda)$ (see (6)) such that $u_{m,a}$ is a stationary solution of (3) for some m, if $\rho > \rho_1$ and $a = a^*$. Also $a^* > u_0$ if $\rho > \rho^*(\lambda)$, $a^*(\rho^*, \lambda) = u_0$, and $a^*(\infty, \lambda) = \infty$.*

Proof. Substituting this function $u_{m,a}$ into (3), we obtain the following necessary and sufficient condition for it to be a solution: either $a = 0$ or

$$M(\rho, k_m) - 2F'(a^2) = 0. \tag{14}$$

The range of $F'(w)$ for $w \ge u_0^2$ includes $[\gamma, \infty)$, which is the range of $F'_c(w)$ for $w \ge u_0^2$. Thus there exists a solution such that

$$F'(a^2) = F'_c(a^2), \tag{15}$$

provided that $M(\rho, k_m) \ge \gamma$, which is true for some m if and only if $\rho \ge \rho^*(\lambda)$.

Moreover, if $u_0 > 0$ so that $F'(u_0^2) = 0$, the quantity $2F'(a^2)$ takes on all negative values in an interval $[-\alpha, 0]$ as a^2 ranges from 0 to u_0^2. Let $\rho_1(\lambda)$ be such that for each $\rho \in (\rho_1, \rho^*)$, $M(\rho, k_m) \in (-\alpha, 0)$ for some $m(\rho)$. Let ρ be a fixed number $< \rho_0^*$. Since $M(\rho, 0) = 0$ and $M(\rho, k) < 0$ for $k > 0$, the function $M(\rho, k_m)$, $m = 1, 2, \ldots$ assumes values in $(-\alpha, 0)$ for λ large enough. Hence it is easy to see that $\lim_{\lambda\to\infty} \rho_1(\lambda) = -\infty$. Since there is a pair $(a,\ k = k_m)$ satisfying (14) for $\rho \in (\rho_1, \rho^*)$, there exist nontrivial exponential solutions for this range of ρ as well. Some are stable, according to a linearized criterion (Sec. 8). However, for large $(\rho_0^* - \rho)$, the wavenumbers k_m are generally small. This is because $M(\rho, k)$ has a maximum at $k = 0$ for such ρ. Therefore these solutions tend to have maximal wavelength.

All this proves the existence part of the lemma. The stated properties of a^* follow from (14).

Lemma 2 *For $\rho \ge \rho^*(\lambda)$, there exists a finite integer $m = m^*(\rho, \lambda)$ which maximizes $M(\rho, k_m)$. It satisfies $\lim_{\rho\downarrow\rho^*} m^*(\rho, \lambda) = m_0 < \infty$. If $\rho \ge \rho^*(\lambda)$ and a satisfies (14) with $m = m^*$, then $u_{m^*,a}$ is a global minimizer of E. If m^* is unique and F is strictly convex, then there is only one global minimizer for this pair (ρ, λ).*

Proof. It follows from **A2** (1) that for any fixed ρ, $\lim_{k\to\infty} M(\rho, k) = -\infty$. Therefore when maximizing $M(\rho, k_m)$ over m, we need consider only a finite number of values of m, namely those for which $M > -\alpha$ for some suitable α. This establishes the first two statements of the lemma.

We shall show that the stated function $u_{m^*,a}$ is a global minimizer of E when $a \geq u_0$. If $u_0 = 0$ and $\rho = \rho^*$, then $a = 0$, $u_{m^*,a} = 0$, and clearly it is a global minimum. We therefore assume that $u_0 > 0$ or $\rho > \rho^*$. In the following, we drop the subscripts on u, let v be any function in $\mathcal{D}(A) \cap L_\infty$, and calculate $E[u + v]$.

We find that

$$\langle A(u+v), u+v\rangle = \langle Au, u\rangle + 2\mathrm{Re}\langle Au, v\rangle + \langle Av, v\rangle, \tag{16}$$

and the same for B. We have chosen the amplitude a so that (15) holds, which means that

$$F(a^2 + w_1) \geq F(a^2) + w_1 F'(a^2)$$

for all w_1. Choosing $w_1 = 2\Re(u\bar{v}) + |v|^2$, we note that $a^2 + w_1 = |u + v|^2$. Hence

$$\int_0^\lambda F(|u+v|^2)dx \geq \lambda F(a^2) + 2F'(a^2)\int_0^\lambda \Re(u\bar{v})\,dx + F'(a^2)\int_0^\lambda |v|^2 dx. \tag{17}$$

Combining these expressions, we obtain

$$E[u+v] \geq E[u] - 2\Re\langle [Au - \rho Bu - 2F'(a^2)u], v\rangle + \tilde{E}[v], \tag{18}$$

where

$$\tilde{E}[v] = E_0[v] + \frac{1}{\lambda}F'(a^2)\int_0^\lambda |v|^2 dx,$$
$$E_0[v] = -\frac{1}{2}\langle Av, v\rangle + \frac{\rho}{2}\langle Bv, v\rangle. \tag{19}$$

The expression $Au - \rho Bu - 2F'(a^2)u$ in the middle term of (18) vanishes because u is a solution of (3). We conclude that u is a global minimizer of E if $\tilde{E}[v] \geq 0$ for all v.

Now we may express $\tilde{E}[v] = -\frac{1}{2}\langle Lv, v\rangle$, where L is the linear self-adjoint operator given by

$$Lv = Av - \rho Bv - 2F'(a^2)v.$$

Therefore $\tilde{E}$ will be positive, as desired, if all eigenvalues μ of L are nonpositive. The eigenvalue equation is

$$Av - \rho Bv - 2F'(a^2)v = \mu v. \tag{20}$$

This may be solved by Fourier expansion over the period interval $(0, \lambda)$. It therefore suffices to look for solutions of the form $v = e^{ik_m x}$, k_m given by (7). We obtain the set of eigenvalues

$$\mu_m = M(\rho, k_m) - 2F'(a^2), \quad m = 1, 2, \ldots \tag{21}$$

We subtract (14) with $m = m^*$ from this to obtain

$$\mu_m = M(\rho, k_m) - M(\rho, k_{m^*}). \tag{22}$$

Therefore since $m = m^*$ was chosen to maximize $M(\rho, k_m)$, we obtain from (22) that all eigenvalues satisfy

$$\mu \leq 0,$$

and hence $E[u + v] \geq E[u]$ for all admissible v. Thus u_{a,m^*} is a global minimizer.

It will be the only one if $E[u + v] > E[u]$ for $v \neq 0$. Suppose now that m^* is unique and F is strictly convex. Again, consider first the case $u \neq 0$. Then $\mu_m < 0$ except for $m = m^*$, so that $E[u + v] > E[u]$ except when $v = cu_{m^*,a}(x)$. We need only verify the inequality in that remaining case, namely $u + v = (a + c)e^{ik_{m^*}x}$, $c \neq 0$, $a + c \geq 0$. By the strict convexity of F, (17) may now be strengthened to

$$\frac{1}{\lambda}\int_0^\lambda F(|u + v|^2)dx = F(|a + c|^2) > F(a^2) + F'(a^2)(2ac + c^2).$$

Hence strict inequality holds in (18), and $E[u+v] > E[u]$. The case $u_{m^*,a} = 0$ can be handled in a similar fashion, replacing $u_{m^*,a}$ above by the eigenfunction of (20) corresponding to μ_{m^*}. This completes the proof.

Remark: Values of m which do not maximize $M(\rho, k_m)$ generate stationary solutions which, although not global minimizers, may be stable. This issue is taken up in Sec. 8.

Lemma 3 *If $\rho \leq \rho^*$, then*

$$E[u] \geq 0 \tag{23}$$

for any u in the domain of E. Moreover if equality holds in (23), then necessarily $|u| \equiv u_0$. If equality holds and $\rho < \rho^*$, then $u =$ const.

Proof. We give the proof only for the case $u_0 > 0$, so that $F'(u_0^2) = 0$. By (4), (19), $E[u] \geq E_0[u]$. The form E_0 is positive if all the eigenvalues of $-A + \rho B$ are nonnegative, i.e. all the eigenvalues of $A - \rho B$ are ≤ 0.

These latter eigenvalues can be found by using exponential functions, and are given by (21) with the last term set equal to zero, i.e. $\mu_m = M(\rho, k_m)$. But when $\rho \leq \rho^*$, this is nonpositive for all m. This completes the proof of the first statement. If equality holds in (23), the last term in (4) vanishes, so that $F(|u|^2) \equiv 0$, hence $|u|^2 = u_0^2$. If moreover $\rho < \rho^*$, then also $\langle Au, u \rangle = 0$ and $u = \text{const.}$ by **A3**.

Proof of Theorem 1

Parts (a) and (c) follow directly from Lemmas 1 and 2, and part (b) from Lemma 3.

Consider now part (d). If it were not true for the minimizers constructed in Lemma 2, then there would exist a sequence $\rho_n \to \infty$ and an integer $\bar{m}$ with $m^*(\rho_n, \lambda) = \bar{m}$ for all n. Since m^* maximizes M, we have that for every $m > \bar{m}$ and every n, $M(\rho_n, k_m) \leq M(\rho_n, k_{\bar{m}})$. Thus from (5), setting $\Delta\hat{A} = \hat{A}(k_{\bar{m}}) - \hat{A}(k_m)$, $\Delta\hat{B} = \hat{B}(k_{\bar{m}}) - \hat{B}(k_m)$, we get

$$\rho_n \Delta\hat{B} \leq \Delta\hat{A}.$$

However, by the condition (10), we can always choose a number m so that $\Delta\hat{B} > 0$. Letting $n \to \infty$ then gives us a contradiction, which proves item (d).

Part (e): From (14), we have

$$a^2 = \phi(M(\rho, k_{m^*})/2).$$

Note that $\phi(t)$ is defined for $t \geq 2F'(u_0^2+)$. Now let m be any fixed integer > 0. We have $M(\rho, k_{m^*})/2 \geq M(\rho, k_m)/2 \equiv C_1 + C_2\rho$, where the C_i depend on k_m and $C_2 = \frac{1}{2}|\hat{B}(k_m)| > 0$. Thus $a^2 \geq \phi(\max[2F'(u_0^2+), C_1 + C_2\rho])$. This yields (11).

Consider now part (f). Let $\rho > \rho_0^*$, and let $I(\rho)$ be the bounded closed interval on the k-axis where $M(\rho, k) \geq 0$.

For all λ, the maximization of $M(\rho, k_m)$ over m is the maximization over a discrete set of values of k in $I(\rho)$, and that in the maximization of $M(\rho, k)$ is taken over all of $I(\rho)$. Therefore the lim sup and lim inf appearing in (12) lie in I, and in fact $\liminf_{\lambda\to\infty} k_{m^*} = \bar{k} > 0$. This proves (12). Finally, (13) follows from this and (14).

Finally, consider part (g). Let $R(s) = \frac{\hat{A}(s)}{\hat{B}(s)}$. It is positive for $s > 0$. Since it has a strict minimum at $s = 0$, the sequence $\{R(mh), \quad m = 1, 2, \dots\}$ for given $h > 0$ has a strict minimum at $m = 1$, provided that h is small enough. Thus setting $h = \frac{2\pi}{\lambda}$, we have that the infimum in (8) is attained at $m = 1$ for λ large enough. By the above construction, when ρ is near enough to ρ^*, this is the value of m at which the infimum is attained in (6). Thus $m_0 = 1$.

This completes the proof of the theorem.

5 Example: an integrodifferential operator derived from an activator–inhibitor system

As our first example of the foregoing theory, we consider the evolution equation

$$u_t = \delta^2 u_{xx} - \rho(G_\epsilon * u - u) - f(u), \tag{24}$$

involving the convolution with a real function $G_\epsilon(y)$ and real parameters $\epsilon > 0,\ \delta > 0$.

Here

$$G_\epsilon(y) = \frac{1}{\epsilon} G\left(\frac{y}{\epsilon}\right), \tag{25}$$

$$\int_{-\infty}^{\infty} G(y)dy = 1,\ G \geq 0,\ G(-y) = G(y). \tag{26}$$

Then

$$Au = \delta^2 u_{xx}, \qquad Bu = G_\epsilon * u - u. \tag{27}$$

There is a connection between (24) and the reaction-diffusion system

$$u_t = \delta^2 u_{xx} - \rho(v - u) - f(u), \tag{28}$$

$$\epsilon^2 v_{xx} + u - v = 0. \tag{29}$$

This system (also in higher dimensions) was studied in [16] and other places.

If u and v are taken to be real, the reaction terms $\rho(u-v) - f(u)$ (at least for ρ large enough) and $u - v$ are monotone increasing in u and decreasing in v, so that it can be considered an activator-inhibitor system. Since there is no term v_t, the kinetics of the inhibitor v is infinite (meaning very large diffusion and very large reaction rate). The second equation (29) can be solved for v in terms of u by

$$v = G_\epsilon^* * u,$$

where $G^*(y) = \frac{1}{2}e^{-|y|}$.

Substituting this into (28), we obtain (24) in the special case $G = G^*$.

We proceed to show that (24) fits into our general framework. Real-valued minimizers will be considered below in Secs. 9 and 10. We first derive a representation for $\langle Bu, u\rangle$. For any periodic function $w(y)$ with period λ, we have the representation

$$\int_{-\infty}^{\infty} G_\epsilon(y)w(y)dy = \sum_{n=-\infty}^{\infty}\int_0^\lambda G_\epsilon(y+n\lambda)w(y)dy = \int_0^\lambda H_\epsilon(y)w(y)dy, \quad (30)$$

where

$$H_\epsilon(y) = \sum_{n=-\infty}^{\infty} G_\epsilon(y+n\lambda). \quad (31)$$

Note that

$$\int_0^\lambda H_\epsilon(y)dy = 1,\ H_\epsilon(-y) = H_\epsilon(y),\ H_\epsilon(y) \geq 0. \quad (32)$$

Therefore

$$\langle G_\epsilon * u - u, u\rangle = \int_0^\lambda\int_0^\lambda H_\epsilon(y-x)(u(y)-u(x))\bar{u}(x)dxdy. \quad (33)$$

Interchanging x and y and using the evenness of H_ϵ, we also have

$$\langle G_\epsilon * u - u, u\rangle = \int_0^\lambda\int_0^\lambda H_\epsilon(y-x)(u(x)-u(y))\bar{u}(y)dxdy$$

$$= -\int_0^\lambda\int_0^\lambda H_\epsilon(y-x)(u(y)-u(x))\bar{u}(y)dxdy.$$

Adding this to (33), we find

$$\langle Bu, u\rangle = \langle G_\epsilon * u - u, u\rangle = -\frac{1}{2}\int_0^\lambda\int_0^\lambda H_\epsilon(y-x)|u(x)-u(y)|^2dxdy =$$

$$= -\frac{1}{2}\int_{-\infty}^{\infty}\int_0^\lambda G_\epsilon(y)|u(x+y)-u(x)|^2dxdy.$$

Clearly $\langle Au, u\rangle = -\delta^2\int_0^\lambda |u_x|^2dx$.

Thus in this example,

$$E[u] = \frac{1}{\lambda}\int_0^\lambda \frac{1}{2}\delta^2|u'|^2dx - \frac{1}{\lambda}\frac{\rho}{4}\int_{-\infty}^{\infty}\int_0^\lambda G_\epsilon(y)|u(x+y)-u(x)|^2dxdy + \frac{1}{\lambda}\int_0^\lambda F(|u|^2)dx. \quad (34)$$

To apply the general theory, we note that

$$\hat{A}(k) = -\delta^2 k^2, \quad \hat{B}(k) = \hat{G}(\epsilon k) - 1,$$

where $\hat{G}(k) = \int_{-\infty}^{\infty} G(y) e^{iky} dy$ is $\sqrt{2\pi}$ times the Fourier transform of G.

Theorem 2 *In the case of the equation (24) under (25), (26), we have that (10) holds, as do all the conclusions of Thm. 1. There exists a number Λ depending only on G such that $m_0 = 1$ in (9) if $\frac{\lambda}{\epsilon} > \Lambda$. In the case of the system (28), (29), the expressions "liminf" and "limsup" in (12) and (13) may be replaced by "lim", and*

$$\bar{k}(\rho) = \frac{1}{\epsilon}\left[\frac{\rho^{1/2}\epsilon}{\delta} - 1\right]^{1/2}. \tag{35}$$

Proof We verify assumptions **A1**–**A3**, as well as (10). The operators A and B are negative, because the corresponding quadratic forms given by the integrals in the first two terms of (34) are positive. Moreover, those forms vanish only for $u = \text{const}$, so that $\hat{A} < 0$, $\hat{B} < 0$ for $k > 0$. This establishes **A1** and **A3**.

Since $\hat{G}(\infty) = 0$, we have $\hat{B}(\infty) = -1$, so that **A2** holds, as well as (10).

To verify that $m_0 = 1$ in (9), according to part (g), we need to show that $\frac{\hat{A}(k)}{\hat{B}(k)}$ has a strict minimum at $k = 0$, i.e. that $p(k) \equiv \frac{1-\bar{G}(k)}{k^2}$ has a strict maximum (which could be infinite) at $k = 0$. An examination of the proof of Thm. 1(g) shows in fact that we need only require $\frac{\lambda}{\epsilon} > \Lambda$ in this case, where Λ is a number independent of δ.

Lemma 4 *The function $p(k)$ takes on a strict maximum at $k = 0$.*

Proof. If $p(k) \to \infty$ as $k \to 0$, the assertion is trivially true. Therefore assume $p(0) < \infty$. Observe that because G is even,

$$\hat{G}(k) = \int_{-\infty}^{\infty} G(y) \cos ky \, dy \tag{36}$$

We also have for $k > 0$

$$\begin{aligned}
\int_{-\infty}^{\infty} G(y)(1 - \cos ky)\, dy &= \frac{1}{k}\int_{-\infty}^{\infty} G(s/k)(1 - \cos s)\, ds \\
&< \frac{1}{2k}\int_{-\infty}^{\infty} G(s/k)s^2\, ds \\
&= \frac{k^2}{2}\int_{-\infty}^{\infty} G(y)y^2\, dy
\end{aligned}$$

hence

$$p(k) < \frac{1}{2}\int_{-\infty}^{\infty} G(y)y^2\,dy = p(0) \tag{37}$$

The proof of the Lemma is now complete.

The conclusion that $m_0 = 1$ follows by using this lemma to verify the condition for part (g) of Theorem 1.

In the particular case $G(y) = \frac{1}{2}e^{-|y|}$, i.e. for (28), (29), we calculate

$$M(\rho, k) = -\delta^2 k^2 + \rho\frac{\epsilon^2 k^2}{1+\epsilon^2 k^2},$$

(35) holds,

$$\rho_0^* = \frac{\delta^2}{\epsilon^2},$$

and since $M(\rho, k)$ has a unique maximum $\bar{k}$, it follows from the proof of (f) that $\lim_{\lambda\to\infty} k(\rho, \lambda) = \bar{k}$. This completes the proof of the theorem.

A few comments are in order about the relevance of activator-inhibitor systems in biology. In the early 1970's, Gierer and Meinhardt [15] introduced activator-inhibitor models in the course of their studies of pattern formation in primitive organisms. Their concept spawned a great deal of research in both biology and mathematics. In typical cases, activator-inhibitor systems are semilinear coupled pairs of parabolic equations governing the production and dispersion of two hypothetical substances: an activator u and an inhibitor v. The production terms of both equations are monotone increasing in u and decreasing in v, at least for a range of values of (u, v) which contains a constant stationary solution. The substance v diffuses more rapidly than does u.

A rough intuitive description of the basis of these models is this. Consider a solution of the evolution system which is constant in both space and time. If this uniform distribution is perturbed near a single location by the introduction, say, of an extra amount of u, this additional activator will cause both u and v to be produced at that location in greater amounts. The inhibitor v, however, diffuses away rapidly, spreading its effect over a larger territory. Its inhibiting effect is thereby diminished at the original point of production. As a result, the initial surplus of u can continue to increase locally to form a spike-like inhomogeneity, or pattern. Nonlinear effects prevent it from becoming too large, and a stable spatial pattern results.

In short, unequal diffusivities may cause the uniform distribution to be unstable, with typical instabilities growing to form spatially patterned states.

The example (28), (29) in this section is such an activator-inhibitor system in which the inhibitor (v) has very fast kinetics, so that its time derivative is missing from the equation.

It should be strongly emphasized that the basic issue here is the stability of activator-inhibitor patterned solutions, rather than their existence. In fact, the existence of spatially nonuniform steady solutions of nonlinear parabolic systems is a commonplace occurrence, but their stability is a much rarer phenomenon and a more difficult question to resolve. The stability question was considered previously in [13].

6 Example: a fourth order differential equation

Here we take the example

$$Au = -u_{xxxx}, \qquad \hat{A}(k) = -k^4, \tag{38}$$

$$Bu = u_{xx}, \qquad \hat{B}(k) = -k^2. \tag{39}$$

This could be obviously generalized to other negative differential operators A and B, with the order of A greater than that of B.

The assumptions **A1**, **A2**, **A3** and the condition (10) are immediate.

In this example, (3) takes the form

$$u_t = -u_{xxxx} - \rho u_{xx} - f(u), \tag{40}$$

and

$$E[u] = \frac{1}{2\lambda}\int_0^\lambda (u_{xx})^2 dx - \frac{\rho}{2\lambda}\int_0^\lambda (u_x)^2 dx + \frac{1}{\lambda}\int_0^\lambda F(|u|^2)dx. \tag{41}$$

This equation, as well as some generalizations of it, were studied in [18, 7, 26].

The equation (40) is akin to the Swift-Hohenberg equation, which in its simplest form is

$$u_t = -(\nabla^2 + 1)^2 u + \alpha u - u^3. \tag{42}$$

This latter equation, together with its generalizations, have been, and continue to be, extremely popular as models for various kinds of patterns in nature. Many references can be found in [12] and [11]. It reduces to (40) if we set $\rho = 2$, $f(u) = (\alpha - 1)u + u^3$. In usual applications, the control parameter is taken to be α rather than the coefficient of $\nabla^2 u$, as we have it.

All the conclusions of Thm. 1 hold for (40), including the hypothesis and conclusion of part (g).

We have $M(\rho, k) = k^2(\rho - k^2)$, $\rho^*(\lambda) = \frac{4\pi^2}{\lambda^2}$, $\rho_0^* = 0$, $\bar{k}(\rho) = \sqrt{\rho/2}$, $m_0 = 1$.

Thus from (12), for large λ we have $k(\rho, \lambda) \sim \sqrt{\rho/2}$ and $m^* \sim \lambda\sqrt{\frac{\rho}{8\pi^2}}$.

7 Example: A convolution equation with indefinite kernel

Let $G^{\pm}(y)$ be two functions, each satisfying (26). We consider the evolution equation

$$u_t = G_\epsilon^+ * u - u - \rho(G_\epsilon^- * u - u) - f(u). \tag{43}$$

If we set $G = G^+ - \rho G^-$, then this equation takes the form

$$u_t = G_\epsilon * u - Iu - f(u),$$

where $I = \int_{-\infty}^{\infty} G(y)dy$. We therefore have an integrodifferential equation similar to that in [1, 4, 5, 6, 14], but with a kernel which can change sign.

Identifying $Au = G_\epsilon^+ * u - u$, $\quad Bu = G_\epsilon^- * u - u$, we obtain (3). Also

$$\hat{A}(k) = \hat{G}^+(\epsilon k) - 1, \quad \hat{B}(k) = \hat{G}^-(\epsilon k) - 1, \tag{44}$$

Leaving aside the question of global solvability of the initial value problem for (43), we concentrate on the possible global minimizers of the associated energy (4).

Although assumptions **A1** and **A3** are satisfied, **A2** is not; in fact $\frac{\hat{A}(k)}{\hat{B}(k)} \to 1$.

The following theorem applies to a general situation suggested by this example.

We consider two cases (one may apply at one value of $\rho > \rho^*(\lambda)$, and the other at a different value of ρ):

(i) $M(\rho, k_m)$ has a maximum with respect to m at a finite value $m^*(\rho)$.

(ii) $M(\rho, k)$ approaches its supremum with respect to k only as $k \to \infty$.

Theorem 3 *Assume that* **A1** *and* **A3** *hold, and*

$$\lim_{k\to\infty} \frac{\hat{A}(k)}{\hat{B}(k)} \to 1. \tag{45}$$

If ρ is such that Case (i) holds, then the applicable conclusions of Thm. 1 hold for that ρ.

On the other hand if Case (ii) holds, then there is no global minimizer of exponential form for that value of ρ. However there is a minimizing sequence of exact stationary solutions of (3) of exponential form, along which the wavenumbers approach ∞. The energy levels of these solutions approach a finite limit, as do their amplitudes.

The proof is along the lines of the foregoing, and will be omitted. If $M(\rho, k)$ has its supremum only at $k = \infty$ there exist infinite sequences $\{k_m\}$ (see (7)) and a_m, defined for sufficiently large m, which satisfy (14). They generate exponential solutions whose amplitudes approach a solution of (14) with $k = \infty$. As $m \to \infty$, they are ever more finely oscillatory. Although they are not minimizers, their energies are ever closer to the infimum of $E[u]$.

8 Linear stability of solutions which are not necessarily minimizers

Linear and weakly nonlinear stability analyses of spatially periodic solutions of nonlinear partial differential equations, including steady solutions and traveling waves, have been the object of many investigations in the past (see for example the excellent survey [12]). In particular, Newton and Keller [27, 28] considered a wide class of equations and systems for which there exist sinusoidal (complex exponential) solutions. They showed that a linear stability analysis of such systems leads generally to an algebraic dispersion relation for the linear growth rate as a function of the wave number and amplitude of the original solution, and the wave number of the perturbation. This in turn leads to a stability criterion. Their results, when applied to the real Ginzburg-Landau equation, provide the classical Eckhaus instabilities; the authors applied them also to a wide variety of other models from mathematical physics.

Their class of problems includes much more than just gradient flows, which we have discussed here. Nominally for differential equations, their method is nevertheless applicable to our equation (3). Here we describe the method and the results of the linearized stability analysis for our equations (3). However, we emphasize that this analysis is for solutions on the whole real line.

The basic focus of the work of Newton-Keller is on systems of differential equations of the form

$$F(i\partial_t, -i\partial_x, |u|^2)u = 0, \tag{46}$$

which generally admit solutions of the form

$$u_0(x,t) = aRe^{i(kx-\omega t)} \tag{47}$$

for some $a,\ R,\ k,\ \omega$.

The stability analysis is with regard to solutions on the entire real line, rather than on a finite period interval, as in our context. It is found to be convenient, and no restriction,to write the perturbed solutions in the form

$$u(x,t;\epsilon) = u_0(x,t) + \epsilon e^{i(kx-\omega t)}\phi(x,t) + o(\epsilon). \tag{48}$$

In our case, the linearization of (3) about

$$u_0(x) = ae^{ikx} \tag{49}$$

takes the form

$$\phi_t = A(\phi e^{ikx})e^{-ikx} - \rho B(\phi e^{ikx})e^{-ikx} - \phi F'(a^2) - 2a^2\Re\phi F''(a^2). \tag{50}$$

Seeking solutions of this equation in the form

$$\phi(x,t) = C_1 e^{i\ell x+\sigma t} + C_2 e^{-i\ell x+\sigma^* t}, \tag{51}$$

and recalling that (14) must hold, we find the following pair of equations for C_1 and $\bar{C}_2$:

$$\begin{array}{l} C_1[\sigma - M(\rho, k+\ell) + F' + a^2F''] + a^2f''\bar{C}_2 = 0, \\ \bar{C}_2[\sigma - M(\rho, k-\ell) + F' + a^2F''] + a^2f''C_1 = 0, \end{array} , \tag{52}$$

where the functions $F',\ F''$ are evaluated at a^2. The determinant condition for the existence of nontrivial solutions leads to the following second order equation for σ, where we set $M_\pm(k,\ell) = M(\rho,k) - M(\rho,k\pm\ell)$:

$$\sigma^2 + \sigma(2a^2F'' + M_+ + M_-) + M_+M_- + a^2F''(M_+ + M_-) = 0. \tag{53}$$

A necessary and sufficient condition on the coefficients of this equation can easily be written for the larger of the real parts of the two roots to be

≤ 0. This leads to the following

Criterion for linear stability of (49):

$$4a^2F''(a^2) + M_+ + M_- \geq 0 \quad \text{and} \quad -2a^2F''(M_+ + M_-) \leq M_+M_- \qquad (54)$$

for all choices of ℓ.

Proposition 1 *If $F''(a^2) < 0$, then u_0 is unstable according to this criterion. If $F''(a^2) = 0$, then u_0 is stable only if the function $M(\rho, k)$ attains a maximum at the given value k.*

Proof. Let $\Sigma(k, \ell) = M_+(k, \ell) + M_-(k, \ell)$. Since $\lim_{\ell \to 0} \Sigma(k, \ell) = 0$, we can find values of ℓ for which the first inequality of this criterion fails if $F''(a^2) < 0$. This proves the first statement. If $F''(a^2) = 0$, then according to the above criterion, stability would imply that $M_+ + M_-$ and M_+M_- are both ≥ 0 for all ℓ. The first in turn implies that the tangent line to the graph of $M(\rho, k)$ (as a function of k) at the given point k lies everywhere above the graph (or touches it). The second implies (by considering arbitrarily small ℓ) that this tangent line is horizontal. This establishes the proposition.

An important case is when the index k and the function $M(\rho, k)$ is such that $M_+ + M_- > 0$ for all $\ell \neq 0$ (because of assumption **A2**, this inequality always holds for large enough ℓ). This is true if the concave hull of the graph of the function $M(\rho, k)$ touches the graph itself at the chosen point k and $M_{kk} < 0$ there. Then from (54) we obtain a sufficient condition for stability involving a function $R(\rho, k)$, defined simply in terms of the function M and number k:

$$R(\rho, k) = \sup_{\ell} \frac{-(M(\rho, k) - M(\rho, k+\ell))(M(\rho, k) - M(\rho, k-\ell))}{(M(\rho, k) - M(\rho, k+\ell)) + (M(\rho, k) - M(\rho, k-\ell))}. \qquad (55)$$

Then the criterion for stability is

$$R(\rho, k) \leq 2a^2F''. \qquad (56)$$

The criterion is easily modified to handle problems in a finite period interval λ, as we were doing. Then the numbers ℓ are simply restricted to be multiples of $\frac{2\pi}{\lambda}$. All the above continues to hold.

In the case of the global minimizers we have been considering, k has been taken to maximize the function $M(\rho, k)$ under the restriction that k be of the form (7), and also $F''(a^2) \geq 0$. Therefore the numerator of (55) is nonpositive and the denominator nonnegative, so that the condition (56) clearly holds, as expected.

But (56) also indicates that other solutions may be stable but not global minimizers. In fact when $F'' > 0$ values of k which are near but not at the maximizer for M may satisfy the condition. Secondly, some of the exponential solutions discussed in Lemma 1 for $\rho < \rho^*$ may satisfy this criterion.

II. Restriction to real valued functions

In this part, we remove several restrictions imposed in the preceding theory. We now take as admissibility class for the minimizers the set of real-valued functions which are λ-periodic. We allow nonlinearities F which are not necessarily functions of $|u|^2$ alone, and finally we extend the theory to higher space dimensions.

In the N-dimensional context, the operators A and B act on λ-periodic functions $u(x)$, where now $x = (x_1, x_2, \dots x_N)$ and $\lambda = (\lambda_1, \lambda_2, \dots \lambda_N)$. The assumptions **A1** to **A3** of Sec. 2 are still assumed, with the obvious notational changes: $k = (k_1, k_2, \dots, k_N)$, e^{ikx} means $e^{ik\cdot x}$, and the limit in (1) is taken as $|k| \to \infty$. Let $\mathcal{D}_r$ be the set of real-valued functions in $\mathcal{D}(A)$.

We restrict u to be real-valued, and consider now energy functionals of the form

$$E[u] = -\frac{1}{2}\langle Au, u\rangle + \frac{\rho}{2}\langle Bu, u\rangle + \frac{1}{|\Lambda|}\int_\Lambda H(u)dx, \tag{57}$$

where the real C^1 function H has a minimum of 0 at some value $u = u_0$ (if it attains this minimum at more than one point, let u_0 be the maximal one). The integral in (57) is the integral over one period cell Λ.

We also assume that H grows superquadratically as $|u| \to \infty$:

$$\lim_{|u|\to\infty} \frac{H(u)}{u^2} = \infty. \tag{58}$$

The corresponding evolution problem is

$$u_t = Au - \rho Bu - H'(u), \tag{59}$$

and the minimizers ϕ of E in $\mathcal{D}_r \cap L^\infty$ are stable λ-periodic solutions of

$$A\phi - \rho B\phi - H'(\phi) = 0. \tag{60}$$

The major difficulty arising in the real valued case is that it is in general impossible to derive the minimizers explicitly. One consequence of this is that the linear stability analysis of Sec. 8 is not applicable to the non-sinusoidal patterns considered in this part; we therefore restrict attention completely to global minimizers of the energy.

Another consequence is that the existence of minimizers is not always clear. In the sequel we shall assume that for each ρ, λ there is $\phi^\rho \in \mathcal{D}_r(A) \cap L_\infty$ such that

$$\min_{u \in \mathcal{D}_r(A) \cap L_\infty} E[u] = E[\phi^\rho] \tag{61}$$

Remark

In general condition (61) might be difficult to verify without making additional assumptions on the operators A, B. On the other hand in special cases proving (61) usually involves checking that E is coercive and weakly lower semicontinuous. Below we consider the real valued version of the model problem studied in Section 5, so that A and B are given by (27).

We set χ_D to be the characteristic function of the set D. For each $K > 0$ there exist constants $C_1(K, \rho)$, $C_2(\rho)$ such that

$$\frac{\rho}{4} \int_{-\infty}^{\infty} \int_0^\lambda G_\epsilon(y) |u(x+y) - (y)|^2 \, dx dy \leq C_1(\rho, K) + C_2(\rho) \int_0^\lambda u^2(x) \chi_{\{u^2 > K\}} \, dx \tag{62}$$

On the other hand

$$\int_0^\lambda F(u^2) \geq \int_0^\lambda F(u^2) \chi_{\{u^2 > K\}}. \tag{63}$$

Using the fact that $F'(w) \to \infty$ as $w \to \infty$ we conclude that for sufficiently large K we have $[F(u^2 - u^2]\chi_{\{u^2>K\}} > 0$ and thus combining (62), (63) we see that there exists a constant $C_3(K, \rho)$ such that

$$E[u] \geq \frac{1}{\lambda} \left[\int_0^\lambda \frac{\delta}{2} |u'|^2 - C_3(K, \rho) \right] \tag{64}$$

and thus E is coercive.

From the embedding $C^\alpha(0, \lambda) \hookrightarrow H^1(0, \lambda)$, $\alpha \in [0, 1)$ we conclude that E is weakly lower semicontinuous with respect to the $H^1_{per}(0, \lambda)$ norm, where $H^1_{per}(0, \lambda)$ denotes the space of λ-periodic H^1 functions.

From (64) and the weak lower semicontinuity we can verify (61) by a fairly standard argument. We observe that the minimizers are in fact smooth.

9 Dependence of minimizers on ρ

Associated with the function H, we define two other functions H^* and H_0.

In accordance with our periodicity constraint, we consider (as in (7)) wavenumber vectors

$$k^m = (k_1^{m_1}, k_2^{m_2}, \dots k_N^{m_N}) \tag{65}$$

with

$$k_i^{m_i} = \frac{2\pi m_i}{\lambda_i}, \tag{66}$$

not all of the m_i vanishing. Let $|m|$ be the number of integers $i \leq N$ such that $m_i \neq 0$.

When k is of this form, it is clear that the integral

$$\frac{1}{|\Lambda|} \int_\Lambda H(a \prod_i \cos(k_i^{m_i} x_i) + b) dx = H^*(a, b, |m|) \tag{67}$$

depends only on a, b, and $|m|$, i.e. how many indexes m_i vanish. This is because when $m_i \neq 0$, the integrand is nontrivially periodic in x_i of period $\frac{\lambda_i}{m_i}$. For example if all the m_i except one vanish, we have that the integral on the left of (67) is equal to $\frac{1}{\lambda_1}\int_0^{\lambda_1} H\left(a\cos\left(\frac{2\pi m_1}{\lambda_1}x_1\right) + b\right) dx_1 = \int_0^1 H(a\cos(2\pi x_1) + b)\, dx_1$, and if all but two vanish, it is

$$\int_0^1 \int_0^1 H(a \cos(2\pi x_1) \cos(2\pi x_2) + b)\, dx_1 dx_2.$$

In any case, the function H^* is even in a and has a minimum of 0, attained at $(a, b) = (0, u_0)$.

We now set

$$H_0(a, |m|) = \min_b H^*(a, b, |m|), \tag{68}$$

with the minimum attained at a value $b = b^*(a, |m|)$.

Let

$$M^*(|m|) = \inf_{a>0} \frac{4H_0(a, |m|)}{a^2}, \tag{69}$$

which is either attained at a positive value a^* of a (finite because of the superquadratic growth of H) or approached as $a \to 0$ (in which case we set $a^* = 0$).

It is important to emphasize that M^* depends only on the nonlinearity H and $|m|$. For example, in the case $H(u) = (1 - u^2)^2$, it can be readily calculated that $M^* = .899$. Essentially this same calculation, leading to the same sufficient condition for the existence of stable patterns, was done in the context of the equation (40) above by Mizel, Peletier, and Troy [26].

Recalling the definition (65) of k^m, we define

$$\bar{\rho}(\lambda) = \inf\{\rho : M(\rho, k^m) > M^*(|m|) \text{ for some } m\} \tag{70}$$

and

$$\rho_0^*(\lambda) = \inf\{\rho : M(\rho, k^m) > 0 \text{ for some } m\}. \tag{71}$$

Note that if $H(u) = F(u^2)$ and $u_0 > 0$, the number ρ_0^* coincides with ρ^* given by (6).

Theorem 4 *There exists a number $\rho_c(\lambda) \in [\rho_0^*(\lambda), \bar{\rho}(\lambda)]$ such that*

(a) for each $\rho < \rho_c$, there exists no nontrivial global minimizer of the functional E (57) in the class of real-valued functions;

(b) for each $\rho > \rho_c$, there exists such a nontrivial real global minimizer ϕ^ρ of E in $\mathcal{D}_r \cap L^\infty$ with

$$E[\phi^\rho] < 0. \tag{72}$$

Let $E_1(\rho) = -\frac{1}{4}(a^*)^2[M(\rho, k^m) - M^*(|m|)]$, where m is chosen and fixed so that $M(\rho, k^m) > 0$ for some $\rho = \rho_0$. Since M is an increasing function of ρ, this will be true for all $\rho > \rho_0$ as well. For $\sigma > 0$, let

$$P(\sigma) = \max_{|u|\le\sigma}(uH'(u) - 2H(u)).$$

It follows from (58) that $P(\infty) = \infty$; in fact if $uH'(u) - 2H(u)$ were bounded for large u by some number K, then integrating the inequality $uH'(u) - 2H(u) < K$ would imply that $\frac{H(u)}{u^2}$ is bounded.

Theorem 5 *For $\rho > \rho_c$, let ϕ^ρ be a minimizer of E, and $a(\rho) = \max|\phi^\rho(x)|$. Then*

$$P(a(\rho)) \ge -2E_1(\rho). \tag{73}$$

Since P and $-E_1$ are both increasing functions of ρ and $-E_1$ is unbounded, we see that (73) provides a lower bound on the amplitude which grows toward ∞ as $\rho\to\infty$.

Corollary 1 *Assume $H(u) = c|u|^r + O(|u|^{r-1})$ as $u \to \infty$, $r > 2$, and that the corresponding differentiated relation $H'(u) = rcu|u|^{r-2} + O(|u|^{r-2})$ holds. Then*

$$a(\rho) \geq C\rho^{1/r} \tag{74}$$

for large ρ, where C depends only on the function H.

Corollary 2 *In the case $H(u) = (1-u^2)^2$, we have, for $\rho > \rho_c$,*

$$\max_x |\phi^\rho(x)| \geq 1. \tag{75}$$

Proof of Theorem 4

Any global minimizer ϕ^ρ with $E[\phi^\rho] < 0$ must be nontrivial, because constants have $E \geq 0$. To emphasize dependence on ρ, we write $E[u] = E_\rho[u]$. Let

$$\rho_c = \inf\{\rho : E_\rho \text{ has a minimizer } \phi^\rho \text{ with } E_\rho[\phi^\rho] < 0\}.$$

The assertion (b) holds by virtue of this definition and the fact that $E_\rho[\phi]$ is a decreasing function of ρ.

Now suppose that for some number $\rho_0 < \rho_c$, E_{ρ_0} has a nontrivial global minimizer ϕ^{ρ_0}. Then $E_{\rho_0}[\phi^{\rho_0}] = 0$, since the minimum of E_ρ for every ρ is always nonpositive. Since ϕ^{ρ_0} is nontrivial, we have from Assumption **A3** that $\langle B\phi^{\rho_0}, \phi^{\rho_0}\rangle < 0$. Hence $E_\rho[\phi^{\rho_0}]$ is strictly decreasing in ρ, so that $E_\rho[\phi^{\rho_0}] < 0$ for $\rho_0 < \rho < \rho_c$, contradicting the definition of ρ_c. Thus part (a) follows.

We show that ρ_c lies in the indicated interval. If $\rho < \rho_0^*$, $M(\rho, k^m) < 0$ for all m, so that the operator $A - \rho B$ is negative definite, and since $H(u) > 0$, it follows from (57) that $E[u] \geq 0$ for all u, hence by our definition of ρ_c, $\rho \leq \rho_c$, and we conclude that $\rho_c \geq \rho_0^*$.

To show that $\rho_c \leq \bar{\rho}$, we choose $a = a^*$ (69) and $b = b^*(a^*)$ (68) to obtain from (57), (69) that

$$E\left[a^* \prod_i \cos(k_i^{m_i} x) + b^*] = -\frac{1}{4}(a^*)^2[M(\rho, k^m) - M^*(|m|)\right] < 0 \tag{76}$$

for some m for $\rho > \bar{\rho}$. Hence the minimizer ϕ^ρ for such ρ has negative energy and must be nontrivial, so that $\rho \geq \rho_c$.

Proof of Theorem 5:

Since ϕ^ρ is a minimizer, it satisfies (60). Take the scalar product of (60) with ϕ^ρ:

$$\langle (A-\rho B)\phi^\rho, \phi^\rho\rangle - \frac{1}{|\Lambda|}\int_\Lambda \phi^\rho H'(\phi^\rho)dx = 0, \tag{77}$$

so that

$$2E[\phi^\rho] + \frac{1}{|\Lambda|}\int_\Lambda \left(\phi^\rho H'(\phi^\rho) - 2H(\phi^\rho)\right) dx = 0. \tag{78}$$

Note from (76) that $E_1(\rho) = E[a^* \prod_i \cos(k_i^{m_i}x) + b^*]$. Since ϕ^ρ is a minimizer, we have $E[\phi^\rho] \le E_1(\rho)$, hence from (78),

$$2E_1(\rho) + \frac{1}{|\Lambda|}\int_\Lambda \left(\phi^\rho H'(\phi^\rho) - 2H(\phi^\rho)\right) dx \ge 0. \tag{79}$$

Since $P(a(\rho)) \ge \frac{1}{|\Lambda|}\int_\Lambda \left(\phi^\rho H'(\phi^\rho) - 2H(\phi^\rho)\right) dx$, we obtain (73).
This completes the proof.

Proof of corollaries
In the case of Cor. 1, we have

$$P(\sigma) = \max_{|u|\le\sigma}\left(rc|u|^r - 2c|u|^r + O(|u|^{r-1})\right) = (r-2)c\sigma^r + O(\sigma^{r-1}) \quad (\sigma\to\infty).$$

We also have $E_1(\rho) \ge c_1\rho - c_2$ for some positive constants c_j. Thus (74) follows easily. We omit the proof of Cor. 2.

10 Bifurcation results

Suppose $N = 1$ and $u_0 = 1$. Again, we consider real-valued steady state solutions of (3). This time we look for solutions bifurcating from the trivial solution $\phi \equiv 1$, ρ being the bifurcation parameter.

For the purpose of this section we shall make the following assumption

A4. There exists $K_0 > 0$ such that

$$-\sum_{m\ge K_0}\frac{1}{\hat{A}(k_m)} < \infty$$

It is easy to check that if $A = \Delta$ then **A4** holds.

Theorem 6 *Let* $\rho_m(\lambda) = \inf\{\rho : \ M(\rho, k_m) > f'(1)\}$.

For each positive integer m, ρ_m *is a bifurcation point of real-valued steady states from the trivial solution* $\phi \equiv 1$. *In the case of (24), the bifurcating curve in* (u, ρ)*-space can be parameterized as* $\gamma_m(s) = (1 + s\cos k_m x) + o(s), \rho_m + \alpha(s)), s \geq 0$ *where*

$$\alpha(0) = \alpha_s(0) = 0, \ \alpha_{ss}(0) = \frac{3f'''(1)}{-4\hat{B}(k_m)} \tag{80}$$

Remark. Other critical points of E could be found for instance by applying one of the standard Calculus of Variations techniques for unstable critical points [32].

Proof of Theorem 6:

Let $Z = L_2 \cap \operatorname{span}\{\cos(k_m x), \ m \geq 0\}$. Note that $\phi \in Z$ if and only if $\phi(x) = \phi(\lambda - x)$. Furthermore we define $X = \mathcal{D}(A) \cap Z \cap C^0(0, \lambda)$. Now X equipped with the norm $\|\phi\|_X^2 = \|A\phi\|^2 + \|B\phi\|^2 + \|\phi\|^2 + \|\phi\|_{C^0}^2$ is a Banach space.

Set

$$\mathcal{F}(\psi, \rho) = (T - \rho B)\psi - h(\psi) \tag{81}$$

where $T\psi = A\psi - f'(1)\psi$, $h(\psi) = f(1 + \psi) - f'(1)\psi$. If $\psi \in X$ has the Fourier expansion

$$\psi = \sum a_m \cos(k_m x) \tag{82}$$

then

$$A\psi = \sum a_m \hat{A}(k_m) \cos(k_m x) \in Z;$$

a similar formula holds for B. For $\psi \in X$ we also have that $h(\psi) \in C^0(0, \lambda)$ hence $h(\psi) \in Z$. Using this and the smoothness of F it follows that $\mathcal{F} \in C^2(X; Z)$. Moreover T, B are bounded linear operators from X into Z. It can be checked easily that $h(0) = Dh(0) = 0$ (using the Frechet derivative D). We shall use the well known theorem of Crandall and Rabinowitz [8] to show the existence of solutions to

$$\mathcal{F}(\psi, \rho) = 0 \tag{83}$$

bifurcating from simple eigenvalues of the pair (T, B) at $\psi = 0, \ \rho = \rho_m$.

Recall that $\rho \in \mathbf{R}$ is a simple eigenvalue of the pair (T, B) if

$$\dim \mathcal{N}(T - \rho B) = 1 = \operatorname{codim} \mathcal{R}(T - \rho B); \tag{84}$$

$$[B\mathcal{N}(T - \rho B)] \oplus \mathcal{R}(T - \rho B) = Z, \tag{85}$$

where $\mathcal{N}, \mathcal{R}$ denote the null space and the range of $T - \rho B$ respectively [10].

Let $\psi \in \mathcal{N}(T - \rho B)$ with Fourier expansion as in (82). We then have

$$\sum a_m[\hat{A}(k_m) - f'(1) - \rho \hat{B}(k_m)] \cos(k_m x) = 0$$

hence

$$\sum a_m[\hat{A}(k_m) - f'(1) - \rho \hat{B}(k_m)] \langle \cos(k_m x), \cos(k_n x) \rangle = 0$$

It follows that $\phi \in \mathcal{N}(T - \rho B)$ if and only if, for some m, $\psi \in \operatorname{span}\{\cos(k_m x)\}$ and $\rho = \rho_m$. Thus $\dim \mathcal{N}(T - \rho B) = 1$.

We fix $n \geq 1$ and $\rho = \rho_n$. We will show that $\operatorname{codim} \mathcal{R}(T - \rho B) = 1$. Let

$$\psi = \sum_{m \neq n} b_m \cos(k_m x)$$

Since A is closed, a function $\eta \in Z$ defined by

$$\eta = \sum_{m \neq n} a_m \cos(k_m x), \qquad a_m = \frac{b_m}{\hat{A}(k_m) - f'(1) - \rho_n \hat{B}(k_m)}$$

solves the equation $(T - \rho B)\eta = \psi$. From A.2 we have for all sufficiently large m

$$|a_m| \leq 2 \frac{|b_m|}{-\hat{A}(k_m)} \leq 2 \frac{\|\psi\|}{-\hat{A}(k_m)},$$

hence using **A4** we conclude that $\eta \in \mathcal{D}(A) \cap C^0(0, \lambda)$. Thus we have $\eta \in X$ and $\operatorname{span}\{\cos(k_m x), m \neq n\} \in \mathcal{R}(T - \rho B)$. On the other hand if $\psi \in \mathcal{R}(T - \rho B)$ then for some $\eta \in X$ we have $\psi = (T - \rho B)\eta \in \operatorname{span}\{\cos(k_m x), m \neq n\}$. It follows that $\operatorname{span}\{\cos(k_m x), m \neq n\} = \mathcal{R}(T - \rho B)$.

This establishes (84). Since $B[\cos(k_m x)] = \hat{B}(k_m) \cos(k_m x)$, (85) holds as well.

The remaining assertions of the theorem can be easily proved by a standard application of the result of Crandall and Rabinowitz [9].

Remark. In general the bifurcating solutions need not to be global. However if T is invertible and

$$T^{-1}B,\ T^{-1}H, \text{ are compact}$$

then the Global Continuation Theorem of Rabinowitz [31] can be applied and we have that either γ_m is an unbounded subset of $X \times \mathbf{R}$ or γ_m contains the point $(1, \rho)$ for some $\rho \neq \rho_m$.

11 Priority of small vs. large patterns

We now have two criteria for the appearance of patterned solutions, when the nonlinearity is bistable with stable zero at (say) $u_0 = 1$: the one arising from Thm. 6, namely $M(\rho, k_m) > f'(1)$ and that arising from Thm. 4, namely $M(\rho, k_m) > M^*$ (in the case $N = 1$). The second is only a sufficient condition, and the first provides bifurcating solutions whose stability would have to be checked.

As the control parameter ρ increases in magnitude, it will be interesting to determine which of these two criteria is first satisfied. This depends simply on the relative magnitudes of M^* and $f'(1)$, which in turn depend only on H. Thus this priority will be independent of A or B.

In the case $H(u) = (1-u^2)^2$, it was indicated following (69) that $M^* = .899\ldots$, whereas $f'(1) = 8$. Therefore in this case, patterns with amplitude around 1 appear much earlier than the solutions bifurcating from 1. The latter will in many cases be local but not global minimizers of the energy. It is also interesting to note that the criterion involving $f'(1)$ can be obtained formally by constructing a number M_1^* analogous to (69), but by holding $b = 1$ throughout.

12 Dependence of the solutions on λ

We now fix $\rho > \rho_c$ and ask how the minimizers depend on λ. Again for simplicity, we restrict to the dimension $N = 1$, although analogous results hold in general. We suppress the dependence on ρ and write the minimizer ϕ as ϕ_λ. We now emphasize the λ-dependence of the energy functional E by writing it as E^λ. Finally, we write $E(\lambda) = E^\lambda[\phi_\lambda]$.

Since the estimate (73) is independent of λ, we know that the amplitudes of our minimizing patterns remain bounded away from 0 as $\lambda \to \infty$. The following result on the minimal energy holds.

Theorem 7 *For every positive number λ_0, the sequence $E(\nu\lambda_0)$ is nonincreasing in the integer ν and approaches a finite negative limit as $\nu \to \infty$.*

Proof. Any λ_0-periodic function u is also a $\nu\lambda_0$-periodic one; moreover, its $\nu\lambda_0$-energy is identical to its λ_0-energy. This is because (a) $\frac{1}{\lambda}\int_0^\lambda H(u)\,dx$ is clearly the same, and (b) if we expand u in Fourier series $u = \sum_{-\infty}^{\infty} u_n e^{i2n\pi x/\lambda_0}$,

$u_{-n} = u_n$, we see, defining $E_0^\lambda[u] = -\frac{1}{2}\langle (A - \rho B)u, u\rangle$, that

$$E_0^{\lambda_0}[u] = -\frac{1}{2}\sum_{n=-\infty}^{\infty} |u_n|^2 M\left(\rho, \frac{2n\pi}{\lambda_0}\right) = -\frac{1}{2}\sum_{m=-\infty}^{\infty} |\tilde{u}_m|^2 M\left(\rho, \frac{2m\pi}{\nu\lambda_0}\right). \tag{86}$$

where $\tilde{u}_m = u_n$ for $m = \nu n$; $\tilde{u}_m = 0$ otherwise. The right side of (86) is $E_0^{\nu\lambda_0}[u]$. It follows that

$$E(\nu\lambda_0) \le E^{\nu\lambda_0}[\phi_{\lambda_0}] = E^{\lambda_0}[\phi_{\lambda_0}] = E(\lambda_0),$$

which proves the monotonicity of $E(\nu\lambda_0)$.

To show that $E(\nu\lambda_0)$ approaches a limit, it suffices to show that $E^\lambda[u]$ is bounded below, independently of λ and u, for fixed ρ. Let $F(u^2)$ be a nonnegative smooth convex function of u^2 satisfying $F(u^2) \le H(u)$ and (2). Let $\hat{E}$ be the associated energy (4). Thus $\hat{E}[u] \le E[u]$ for all u. In Theorem 1 we found the minimizers of $\hat{E}$, among complex-valued periodic functions, explicitly. They are exponential functions. Their energies are verified to be bounded independently of λ. The minimal energies among real-valued functions are no less than they are among complex-valued ones. Therefore the energies $E[u]$ are likewise bounded below. This completes the proof.

Consider now the minimizer ϕ_{λ_0} for any period interval λ_0. Since it is also a solution of (60) with period $\nu\lambda_0$ for any positive integer N, it is a stationary point for $E^{\nu\lambda_0}$ on that larger interval, but is no longer necessarily a minimizer. If λ_0 is large, however, ϕ_{λ_0} is at worst only weakly unstable with respect to the interval $\nu\lambda_0$, in the sense that solutions of the evolution problem starting near ϕ_{λ_0} move away from ϕ_{λ_0}, if at all, only slowly. Specifically, we have the following result about the L^2 norm of the velocity u_t:

Theorem 8 *Consider the evolution (59), where u is required to be a $\nu\lambda_0$-periodic function of x for each time t, and to satisfy the initial condition $u(x,0) = u_0(x)$. Let $\delta(\lambda)$ be any function of λ approaching 0 as $\lambda\to\infty$. There exists a function $\delta_1(\lambda)$ independent of ν with $\lim_{\lambda\to\infty}\delta_1(\lambda) = 0$ such that if $E^{\nu\lambda_0}[u_0] \le E^{\nu\lambda_0}[\phi_{\lambda_0}] + \delta(\lambda_0) = E(\lambda_0) + \delta(\lambda_0)$, then*

$$\int_0^\infty \|u_t(\cdot,t)\|^2 dt \le \delta_1(\lambda_0). \tag{87}$$

Proof. We calculate

$$\frac{d}{dt}E[u(\cdot,t)] = -\langle Au - \rho Bu - H'(u), u_t\rangle = -\|u_t(\cdot,t)\|^2, \tag{88}$$

hence

$$\int_0^\infty \|u_t(\cdot,t)\|^2 dt = E^{\nu\lambda_0}[u_0] - \lim_{t\to\infty} E^{\nu\lambda_0}[u(\cdot,t)] \le E(\lambda_0) + \delta(\lambda_0) - E(\nu\lambda_0).$$

The conclusion (87) follows by the previous theorem 7 and our assumption on δ.

13 Further questions

The most important continuing questions about the minima of E (57) have to do with their dependence on λ.

It may be expected that in many cases the global minimizers, or translates of them, will approach some periodic function as $\lambda\to\infty$ uniformly on bounded intervals. If that is true, then the pattern's properties will be more or less insensitive to the size of the domain in x-space. (Our theorem 5 establishes a lower bound on the amplitude of ϕ^ρ which is independent of λ; but it is conceivable that no sequence of them, as $\lambda\to\infty$, will approach a periodic stationary solution.)

Simple as this concept may be, apparently the most that have been proved in this direction are results applicable to a class of fourth order differential equations with bistable nonlinearities discussed in Sec. 6 above. Leizarowitz and Mizel [18] showed the existence of a real periodic solution which is a minimizer in their sense. Their concept of minimum is not in reference to any prescribed period; their problem is on the whole real line, and nonperiodic bounded functions are in competition for the minimum. These authors do not obtain uniqueness of the minimizer.

An analysis of a corresponding variational problem on an infinite domain with mass constraint (see the references for precise definitions, conditions, and conclusions) was initiated by Coleman, Marcus, and Mizel [7] and continued by Marcus and Zaslavskii [20, 21, 22, 23]. For both the unconstrained and constrained problem, it was proved in [21] that any sequence of minimizers for the corresponding finite domain problem, in which the length of the domain tends to infinity, has a subsequence converging to a minimizer of the infinite domain problem. Rates of convergence are found and much more.

In [22], the authors deal with unconstrained problems and discuss the shape of minimizers on $(0,\infty)$ and in particular the shape of periodic minimizers. All this is done for a wide class of integrands, not just the basic model. In addition they show that generically, in a very precise way, there is uniqueness (up to translation) of the periodic minimizer. This means that by arbitrarily small perturbations of a given integrand one obtains integrands with this uniqueness property.

In a paper under preparation, these same authors obtain the precise behavior of optimal solutions for the unconstrained problem on the half line and show that at infinity each of them converges to a periodic solution.

Motivated by [18], extensive further investigations into solutions of

$$u_t = -u_{xxxx} - \rho u_{xx} + u - u^3 \tag{89}$$

were made by Mizel, Peletier and Troy in [26]. The same concept of variational problem for functions on the whole line was used as in [18]. A specific estimate for a critical value $\rho = \rho_1$ was given, beyond which nontrivial minimizers exist. It coincides with the number $\bar{\rho}$ (70) calculated for this example.

Symmetry properties of the periodic minimizers were proved; for example they must be even with respect to maxima. Upper and lower bounds for the minimal energy $E(\rho)$ were given in terms of ρ; in particular it was found that $E(\rho)$ approaches $-\infty$ like the 4th power of ρ. Finally, the authors constructed global branches of stationary periodic solutions of (89), the parameter again being ρ.

14 Summary

Minimizers ϕ^ρ of (4) or (57) are stable stationary solutions of (3) or (59), in the sense that no solution $u(t)$ of the evolution problem with $u(-\infty) = \phi^\rho;\ u(t) \neq \phi^\rho,\ t > -\infty$ can exist. Moreover when $E[\phi^\rho] < 0$ that minimum cannot be a constant, since constant solutions have nonnegative energy. In this case the minimizers must be nontrivial periodic functions, i.e. stable patterned solutions. We have found (Theorem 1) that in the complex valued case, a necessary and sufficient condition for the existence of these global minimizers with given λ is $\rho > \rho^*(\lambda)$, where $\rho^*(\lambda)$ is given explicitly in terms of $\hat{A}$, $\hat{B}$, and F. It follows easily that if λ is unrestricted, then a necessary and sufficient condition is $\rho > \rho_0^*$ (Thm. 1(f)). The other conclusions in Theorem 1 give us information about the behavior of the minimizer, namely its size and wavelength, as ρ approaches one or the other of its two limiting values $\rho^*(\lambda)$ and ∞. It is also shown that the properties are quite independent of λ as $\lambda \to \infty$.

Besides the global minimizers, there are many other solutions of complex exponential type which satisfy a weaker stability statement, namely that resulting from a linear stability analysis. This criterion is developed and stated succinctly in Sec. 8. This analysis does not apply to the real-valued solutions in Part II.

Threshold results, with ρ^* replaced by ρ_c (Thm. 4), are also true for the real case, but we have only estimates, rather than a precise value, for ρ_c. Real stable patterned solutions bifurcate from the constant solution $u \equiv u_0$ as ρ increases past bifurcation points. It is shown that in the case of the prototypical bistable nonlinearity $F(u^2) = (1-u^2)^2$, these are not the first patterns which emerge as ρ increases; finite amplitude ones are already present at a value ρ^* less than the first bifurcation point.

In both the real and the complex cases, we have also demonstrated the existence of nontrivial periodic solutions of (59) with arbitrarily small wavelength, provided that ρ is large enough. This follows from our analysis simply by taking λ arbitrarily small. If λ is taken to be a small integral part of a fixed large number λ_0, the sinusoidal solutions that we have in the complex case are stable with respect to evolutions in (complex) $L_2(0, \lambda_0)$. However, this stability statement is not in general true in the real case.

The qualitative dependence of the real minimizers on λ for large λ is an important open question. The most that we have been able to show in general is a lower bound on their amplitudes (Thm. 5) and some properties of their energies (Thms. 7 and 8).

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