Coarse Alexander duality and duality groups

Misha Kapovich*
Bruce Kleiner†

November 1, 1999

Abstract

We study discrete group actions on coarse Poincare duality spaces, e.g. acyclic simplicial complexes which admit free cocompact group actions by Poincare duality groups. When G is an $(n-1)$ dimensional duality group and X is a coarse Poincare duality space of formal dimension n, then a free simplicial action $G \curvearrowright X$ determines a collection of "peripheral" subgroups $F_1, \ldots, F_k \subset G$ so that the group pair $(G; F_1, \ldots, F_k)$ is an n-dimensional Poincare duality pair. In particular, if G is a 2-dimensional 1-ended group of type FP_2, and $G \curvearrowright X$ is a free simplicial action on a coarse $PD(3)$ space X, then G contains surface subgroups; if in addition X is simply connected, then we obtain a partial generalization of the Scott/Shalen compact core theorem to the setting of coarse $PD(3)$ spaces.

1. Introduction

In this paper we study simplicial complexes which behave homologically (in the large-scale) like $\mathbb{R}^n$, and discrete group actions on them. Our main objective is a partial generalization of the Scott/Shalen compact core theorem for 3-manifolds ([32], see also [24]) to the setting of Poincare duality spaces of arbitrary dimension. In the one ended case, the compact core theorem says that if X is a contractible 3-manifold and G is a finitely generated one-ended group acting discretely and freely on X, then the quotient X/G contains a compact core – a compact submanifold with (aspherical) incompressible boundary $Q \subset X/G$ so that the inclusion $Q \to X/G$ is a homotopy equivalence. The proof of the compact core theorem relies on standard tools in 3-manifold theory like transversality, which has no appropriate analog in the 3-dimensional coarse Poincare duality space setting, and the Loop Theorem, which has no analog even for manifolds when the dimension is at least 4.

We now formulate our analog of the core theorem. For our purpose, the appropriate substitute for a finitely generated, one-ended, 2-dimensional group will be a *duality group* of dimension $n-1$. We recall [6] that a group G is a k-dimensional

*Supported by NSF grant DMS-96-26633.
†Supported by a Sloan Foundation Fellowship, and NSF grants DMS-95-05175, DMS-96-26911.

duality group if G is of type FP, $H^i(G;\mathbb{Z}G) = 0$ for $i \neq k$, and $H^k(G;\mathbb{Z}G)$ is torsion-free [1]. Examples of duality groups include:

A. Freely indecomposable 2-dimensional groups of type FP_2; for instance, torsion free one-ended 1-relator groups.

B. The fundamental groups of compact aspherical manifolds with incompressible aspherical boundary [6].

C. The product of two duality groups.

D. Torsion free S-arithmetic groups [9].

Instead of 3-dimensional contractible manifolds, we work with a class of metric simplicial complexes which we call "coarse $PD(n)$ spaces". These will be defined in section 6, but we note that important examples include universal covers of closed aspherical n-dimensional PL-manifolds, acyclic complexes X with $H_c^*(X) \simeq H_c^*(\mathbb{R}^n)$ which admit free cocompact simplicial group actions, and uniformly acyclic n-dimensional PL-manifolds with bounded geometry. We recall that an n-dimensional Poincare duality group ($PD(n)$ group) is a duality group G with $H^n(G;\mathbb{Z}G) \simeq \mathbb{Z}$. Our group-theoretic analog for the compact core will an n-dimensional Poincare duality pair ($PD(n)$ pair), i.e. a group pair $(G; F_1, \ldots, F_k)$ whose double with respect to the F_i's is an n-dimensional Poincare duality group, [14]. In this case the "peripheral" subgroups F_i are $PD(n-1)$ groups. See section 3 for more details.

Theorem 1.1. *Let X be a coarse $PD(n)$ space, and let G be an $(n-1)$-dimensional duality group acting discretely and simplicially on X. Then:*

1. G contains subgroups $F_1, \ldots F_k$ (which are canonically defined up to conjugacy by the action $G \curvearrowright X$) so that $(G; \{F_i\})$ is a $PD(n)$ pair.

2. There is a connected G-invariant subcomplex $K \subset X$ so that K/G is compact, the stabilizer of each component of $X - K$ is conjugate to one of the F_i's, and each component of $\overline{X - K}/G$ is one-ended.

Thus, the duality groups G which appear in the above theorem behave homologically like the groups in example B. As far as we know, Theorem 1.1 is new even in the case that $X \simeq \mathbb{R}^n$, when $n \geq 4$. Theorem 1.1 and Lemma 6.2 imply

Corollary 1.2. *Suppose a $PD(n)$ group Γ acts freely, simplicially, and cocompactly on an acyclic complex. Then any $(n-1)$-dimensional duality subgroup $G \subset \Gamma$ contains a finite collection $H_1, \ldots, H_k$ of $PD(n-1)$ subgroups so that the group pair $(G, \{H_i\})$ is a $PD(n)$ pair.*

In the next theorem we obtain a partial generalization of the Scott-Shalen theorem for groups acting on coarse $PD(3)$ spaces.

Theorem 1.3. *Let $G \curvearrowright X$ be a free simplicial action of a 2-dimensional, one-ended group of type FP_2 on a simply connected coarse $PD(3)$ space X. Then there exists a complex Y and a proper homotopy equivalence $f : X/G \to Y$ which is a homeomorphism away from a compact subset, where $Y = Q \cup (E_1 \sqcup \ldots \sqcup E_k)$, and*

1. Q is a finite subcomplex of Y, and $Q \hookrightarrow Y$ is a homotopy equivalence.

[1] We never make use of the last assumption about $H^k(G;\mathbb{Z}G)$ in our paper.

2. The E_i's are disjoint and one-ended. For each i, $S_i := E_i \cap Q$ is a closed aspherical surface, and $S_i \hookrightarrow E_i$ is a homotopy equivalence.

3. Each inclusion $S_i \hookrightarrow Q$ is π_1-injective.

4. $(Q; S_1, \ldots, S_k)$ is a Poincare pair [36]. In particular, Q is a finite Eilenberg-MacLane space for G.

Corollary 1.4. *If G is a group of type FP_2, $dim(G) \leq 2$, and G acts freely simplicially on a coarse $PD(3)$ space, then either G contains a surface group, or G is free.*

Proof. Let $G = F * (*_i G_i)$ be a free product decomposition where F is a finitely generated free group, and each G_i is finitely generated, freely indecomposable, and non-cyclic. Then by Stallings' theorem on ends of groups, each G_i is one-ended, and hence is a 2-dimensional duality group. By Theorem 1.3, each G_i contains surface subgroups. $\square$

We believe that Theorem 1.3 still holds if one relaxes the FP_2 assumption to finite generation, and we conjecture that any finitely generated group which acts freely, simplicially, but not cocompactly, on a coarse $PD(3)$ space is finitely presented. We note that Bestvina and Brady [2] construct 2-dimensional groups which are FP_2 but not finitely presented.

In Proposition 8.11 we prove an analog of the uniqueness theorem for peripheral structure [25, 26] for fundamental groups of acylindrical 3-manifolds with aspherical incompressible boundary.

We were led to Theorems 1.1, 1.3 by our earlier work on hyperbolic groups with one-dimensional boundary [27]; in that paper we conjectured that every torsion-free hyperbolic group G whose boundary is homeomorphic to the Sierpinski carpet is the fundamental group of a compact hyperbolic 3-manifold with totally geodesic boundary. In the same paper we showed that such a group G is part of a canonically defined $PD(3)$ pair and that our conjecture would follow if one knew that G were a 3-manifold group. One approach to proving this is to produce an algebraic counterpart to the Haken hierarchy for Haken 3-manifolds in the context of $PD(3)$ pairs. We say that a $PD(3)$ pair $(G; H_1, \ldots, H_k)$ is *Haken* if it admits a nontrivial splitting[2]. One would like to show that Haken $PD(3)$ pairs always admit nontrivial splittings over $PD(2)$ pairs whose peripheral structure is compatible with that of G. Given this, one can create a hierarchical decomposition of the group G, and try to show that the terminal groups correspond to fundamental groups of 3-manifolds with boundary. The corresponding 3-manifolds might then be glued together along boundary surfaces to yield a 3-manifold with fundamental group G. At the moment, the biggest obstacle in this hierarchy program appears to be the first step; and the two theorems above provide a step toward overcoming it under the assumption that finitely generated subgroups of $PD(3)$ groups are of type FP_2.

As an application of Theorems 1.1 and 1.3 and the techniques used in their proof, we give examples of $(n-1)$-dimensional groups which cannot act discretely and simplicially on coarse $PD(n)$ spaces (see section 9 for details):

[2]If $k > 0$ then such a splitting always exists.

1. A 2-dimensional one-ended group of type FP_2 with positive Euler characteristic cannot act on a coarse $PD(3)$ space. For example, the semi-direct product of two finitely generated free groups.

2. For $i = 1, ..., \ell$ let G_i be a duality group of dimension n_i and assume that for $i = 1, 2$ the group G_i is not a $PD(n_i)$ group. Then the product $G_1 \times ... \times G_\ell$ cannot act on a coarse $PD(n)$ space where $n - 1 = n_1 + ... + n_\ell$. The case when $n = 3$ is due to Kropholler, [28].

3. If G_1 is a k-dimensional duality group and G_2 is the the Baumslag-Solitar group $BS(p, q)$ (where $p \neq \pm q$), then the direct product $G_1 \times G_2$ cannot act on a coarse $PD(3+k)$ space. In particular, $BS(p, q)$ cannot act on a coarse $PD(3)$ space.

4. An $(n-1)$-dimensional group G of type FP_{n-1} which contains infinitely many conjugacy classes of coarsely non-separating maximal $PD(n-1)$ subgroups.

Our theme is related to the problem of finding an n-thickening of an aspherical polyhedron P up to homotopy, i.e. finding a homotopy equivalence $P \to M$ where M is a compact manifold with boundary and $dim(M) = n$. If $k = dim(P)$ then we may immerse P in $\mathbb{R}^{2k}$ by general position, and obtain a $2k$-manifold thickening M by "pulling back" a regular neighborhood. Given an n-thickening $P \to M$ we may construct a free simplicial action of $G = \pi_1(P)$ on a coarse $PD(n)$ space by modifying the geometry of $Int(M)$ and passing to the universal cover. In particular, if G cannot act on a coarse $PD(n)$ space then no such n-thickening can exist. In a subsequent paper with M. Bestvina [3] we give examples of finite k-dimensional aspherical polyhedra P whose fundamental groups cannot act freely simplicially on any coarse $PD(n)$ space for $n < 2k$, and hence the polyhedra P do not admit n-thickening for $n < 2k$.

In this paper we develop and use ideas in coarse topology which originated in earlier work by a number of authors: [8, 20, 21, 23, 29, 30, 31]. Other recent papers involving similar ideas include [34, 18, 19], and especially [10], which has considerable overlap with this paper. An adaptation of Richard Schwartz's coarse Alexander duality to coarse $PD(n)$ spaces plays an important role in the proofs of our main results.

To give an idea of the proof of Theorem 1.1, consider the case when the coarse $PD(n)$-space X happens to be $\mathbb{R}^n$ with a uniformly acyclic bounded geometry triangulation. We take combinatorial tubular neighborhoods $N_R(K)$ of a G-orbit K in X and analyze the structure of connected components of $X - N_R(K)$. Following R. Schwartz we call a connected component C of $X - N_R(K)$ *deep* if C is not contained in any tubular neighborhood of K. When G is a group of type FP_n, using Alexander duality one shows that deep components of $X - N_R(K)$ *stabilize*: there exists R_0 so that no deep component of $X - N_{R_0}(K)$ breaks up into multiple deep components as R increases beyond R_0. If G is an $(n-1)$-dimensional duality group then the idea is to show that the stabilizers of of deep components of $X - N_{R_0}(K)$ are $PD(n-1)$-groups, which is the heart of the proof. These groups define the peripheral subgroups $F_1, \ldots, F_k$ of the $PD(n)$ pair structure $(G; F_1, \ldots, F_k)$ for G.

When X is a coarse $PD(n)$-space rather than $\mathbb{R}^n$, one does not have Alexander duality since Poincare duality need not hold locally. However there is a coarse version of Poincare duality which we use to derive an appropriate coarse analogue of Alexander duality. Roughly speaking this goes as follows. If $K \subset \mathbb{R}^n$ is a subcomplex then

Poincare duality gives an isomorphism

$$H_c^*(K) \to H_{n-*}(\mathbb{R}^n, \mathbb{R}^n - K).$$

This fails when we replace $\mathbb{R}^n$ by a general coarse $PD(n)$ space X. We prove however that for a certain constant D there are homomorphisms defined on tubular neighborhoods of K:

$$P_{R+D} : H_c^k(N_{D+R}(K)) \to H_{n-k}(X, Y_R), \text{ where } Y_R := \overline{X - N_R(K)},$$

which determine an *approximate isomorphism*. This means that for every R there is an R' (one may take $R' = R + 2D$) so that the homorphisms a and b in the following commutative diagram are zero:

$$\begin{array}{ccccccc} ker(P_{R'}) & \to & H_c^k(N_{R'}(K)) & \xrightarrow{P_{R'}} & H_{n-k}(X, Y_{R'-D}) & \to & coker(P_{R'}) \\ a \downarrow & & \downarrow & & \downarrow & & b \downarrow \\ ker(P_R) & \to & H_c^k(N_R(K)) & \xrightarrow{P_R} & H_{n-k}(X, Y_{R-D}) & \to & coker(P_R) \end{array}$$

This coarse version of Poincare duality leads to coarse Alexander duality, which suffices for our purposes.

Organization of the paper. In section 2 we introduce metric simplicial complexes and recall notions from coarse topology. Section 3 reviews some facts and definitions from cohomological group theory, duality groups, and group pairs. In section 4 we define approximate isomorphisms between inverse and direct systems of abelian groups, and compare these with Grothendieck's pro-morphisms. Section 5 provides finiteness criteria for groups, and establishes approximate isomorphisms between group cohomology and cohomologies of nested families of simplicial complexes. In section 6 we define coarse $PD(n)$ spaces, give examples, and prove coarse Poincare duality for coarse $PD(n)$ spaces. In section 7 we prove coarse Alexander duality and apply it to coarse separation. In section 8 we prove Theorems 1.1, 1.3, Proposition 8.11, and variants of Theorem 1.1. In section 9 we apply coarse Alexander duality and Theorem 1.1 to show that certain groups cannot act freely simplicially on coarse $PD(n)$ spaces. In the Appendix (section 10) we give a brief account of coarse Alexander duality for uniformly acyclic triangulated manifolds of bounded geometry. The reader interested in manifolds and not in Poincare comlexes can use this as a replacement of Theorem 7.5.

Suggestions to the reader. Readers familiar with Grothendieck's pro-morphisms may wish to read the second part of section 4, which will allow them to translate statements about approximate isomorphisms into pro-language. Readers who are not already familiar with pro-morphisms may simply skip this. Those who are interested in finiteness properties of groups may find section 5, especially Theorems 5.9 and Corollary 5.12, of independent interest.

Acknowledgements. We are grateful for M. Bestvina and S. Weinberger for useful conversations about coarse Poincare duality.

Contents

2. Geometric Preliminaries

Metric simplicial complexes. Let X be the geometric realization of a connected locally finite simplicial complex. Henceforth we will conflate simplicial complexes with their geometric realizations. We will metrize the 1-skeleton X^1 of X by declaring each edge to have unit length and taking the corresponding path-metric. Such an X with the metric on X^1 will be called a *metric simplicial complex.* The complex X is said to have *bounded geometry* if all links have a uniformly bounded number of simplices; this is equivalent to saying that the metric space X^1 is locally compact and every R-ball in X^1 can be covered by at most $C = C(R,r)$ r-balls for any $r > 0$. In particular, $dim(X) < \infty$. If $K \subset X$ is a subcomplex and r is a positive integer then we define (combinatorial) r-tubular neighborhood $N_r(K)$ of K to be r-fold iterated closed star of K, $St^r(K)$; we declare $N_0(K)$ to be K itself. Note that for $r > 0$, $N_r(K)$ is the closure of its interior. The diameter of K is defined to be the diameter of its zero-skeleton, and ∂K denotes the frontier of K, which is a subcomplex. For each vertex $x \in X$ and $R \in \mathbb{Z}_+$ we let $B(x,R)$ denote $N_R(\{x\})$, the "R-ball centered at x".

Coarse Lipschitz and uniformly proper maps. We recall that a map $f : X \to Y$ between metric spaces is called (L,A)-*Lipschitz* if

$$d(f(x), f(x')) \le Ld(x,x') + A$$

for any $x, x' \in X$. A map is *coarse Lipschitz* if it is (L,A)-Lipschitz for some L, A. A coarse Lipschitz map $f : X \to Y$ is called *uniformly proper* if there is a proper function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ (a *distortion function*) such that

$$d(f(x), f(x')) \ge \phi(d(x,x'))$$

for all $x, x' \in X$.

Throughout the paper we will use simplicial (co)chain complexes and integer coefficients. If $C_*(X)$ is the simplicial chain complex and $A \subset C_*(X)$, then the *support of A*, denoted *Support*(A), is the smallest subcomplex $K \subset X$ so that $A \subset C_*(K)$.

Throughout the paper we will assume that morphisms between simplicial chain complexes preserve the usual augmentation.

If X, Y are metric simplicial complexes as above then a homomorphism

$$h : C_*(X) \to C_*(Y)$$

is said to be *coarse Lipschitz* if for each simplex $\sigma \subset X$, $Support(h(C_*(\sigma)))$ has uniformly bounded diameter. The *Lipschitz constant of* h is

$$\max_{\sigma} diam(Support(h(C_*(\sigma)))).$$

A homomorphism h is said to be *uniformly proper* if it is coarse Lipschitz and there exists a proper function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ (a *distortion function*) such that for each subcomplex $K \subset X$ of diameter $\geq r$, $Support(h(C_*(K)))$ has diameter $\geq \phi(r)$. We will apply this definition only to chain mappings and chain homotopies[3]. We say that a homomorphism $h : C_*(X) \to C_*(X)$ has displacement $\leq D$ if for every simplex $\sigma \subset X$, $Support(h(C_*(\sigma))) \subset N_D(\sigma)$.

We may adapt all of the definitions from the previous paragraph to mappings between other (co)chain complexes associated with metric simplicial complexes, such as the compactly supported cochain complex $C_c^*(X)$.

Coarse topology. A metric simplicial complex X is said to be *uniformly acyclic* if for every R_1 there is an R_2 such that for each subcomplex $K \subset X$ of diameter $\leq R_1$ the inclusion $K \to N_{R_2}(K)$ induces zero on reduced homology groups. Such a function $R_2 = R_2(R_1)$ will be called an *acyclicity function* for $C_*(X)$. Let $C_c^*(X)$ denote the complex of simplicial cochains, and suppose $\alpha : C_c^n(X) \to \mathbb{Z}$ is an augmentation for $C_c^*(X)$. Then the pair $(C_c^*(X), \alpha)$ is *uniformly acyclic* if there is an $R_0 > 0$ and a function $R_2 = R_2(R_1)$ so that for all $x \in X^0$ and all $R_1 \geq R_0$,

$$Im(H_c^*(X, \overline{X - B(x, R_1)}) \to H_c^*(X, \overline{X - B(x, R_2)}))$$

maps isomorphically onto $H_c^*(X)$ under $H_c^*(X, \overline{X - B(x, R_2)}) \to H_c^*(X)$, and α induces an isomorphism $\bar{\alpha} : H_c^n(X) \to \mathbb{Z}$.

Let $K \subset X$ be a subcomplex of a metric simplicial complex X. For every $R \geq 0$, we say that an element $c \in H_k(X - N_R(K))$ is *deep* if it lies in $Im(H_k(X - N_{R'}(K)) \to H_k(X - N_R(K)))$ for every $R' \geq R$; equivalently, c is deep if belongs to the image of

$$\lim_{\substack{\longleftarrow \\ r}} H_k(X - N_r(K)) \longrightarrow H_k(X - N_R(K)).$$

We let $H_k^{Deep}(X - N_R(K))$ denote the subgroup of deep homology classes of $X - N_R(K)$. Hence we obtain an inverse system $\{H_k^{Deep}(X - N_R(K))\}$. We say that the deep homology *stabilizes* at R_0 if the projection homomorphism

$$\lim_{\substack{\longleftarrow \\ R}} H_k^{Deep}(X - N_R(K)) \to H_k^{Deep}(X - N_{R_0}(K))$$

is injective.

[3]Recall that there is a standard way to triangulate the product $\Delta^k \times [0, 1]$; we can use this to triangulate $X \times [0, 1]$ and hence view it as a metric simplicial complex.

Specializing the above definition to the case $k = 0$, we arrive at the definition of deep complementary components. If $R \geq 0$, a component C of $X - N_R(K)$ is called *deep* if it is not contained within a finite neighborhood of K. A subcomplex K *coarsely separates* X if there is an R so that $X - N_R(K)$ has at least two deep components. A deep component C of $X - N_R(K)$ is said to be *stable* if for each $R' \geq R$ the component C meets exactly one deep component of $X - N_{R'}(K)$. K is said to *coarsely separate* X into (exactly) m components if there is an R so that $X - N_R(K)$ consists of exactly m stable deep components.

Note that $H_0^{Deep}(X - N_R(K))$ is freely generated by elements corresponding to deep components of $X - N_R(K)$. The deep homology $H_0^{Deep}(X - N_R(K))$ stabilizes at R_0 if and only if all deep components of $X - N_{R_0}(K)$ are stable.

If $G \curvearrowright X$ is a simplicial action of a group on a metric simplicial complex, then one orbit $G(x)$ coarsely separates X if and only if every G-orbit coarsely separates X; hence we may simply say that G coarsely separates X. If H is a subgroup of a finitely generated group G, then we say that H coarsely separates G if H coarsely separates some (and hence any) Cayley graph of G.

Let Y, K be subcomplexes of a metric simplicial complex X. We say that Y coarsely separates K in X if there is $R > 0$ and two distinct components $C_1, C_2 \subset X - N_R(Y)$ so that the distance function $d_Y(\cdot) := d(\cdot, Y)$ is unbounded on both $K \cap C_1$ and $K \cap C_2$. The subcomplex Y will coarsely separate X in this case.

3. Group theoretic preliminaries

Resolutions, cohomology and relative cohomology. Let G be group and K be an Eilenberg-MacLane space for G. If $\mathcal{M}$ is a system of local coefficients on K, then we have homology and cohomology groups of K with coefficients in $\mathcal{M}$: $H_*(K; \mathcal{M})$ and $H^*(K; \mathcal{M})$. Now let A be a $\mathbb{Z}G$-module. We recall that a *resolution of* A is an exact sequence of $\mathbb{Z}G$-modules:

$$\ldots \to P_n \to \ldots \to P_0 \to A \to 0.$$

Every $\mathbb{Z}G$-module has a unique projective resolution up to chain homotopy equivalence. If M is a $\mathbb{Z}G$-module, then the *cohomology of G with coefficients in M, $H^*(G; M)$,* is defined as the homology of chain complex $Hom_{\mathbb{Z}G}(P_*, M)$ where P_* is a projective resolution of the trivial $\mathbb{Z}G$-module $\mathbb{Z}$; the *homology of G with coefficients in M, $H_*(G; M)$,* is the homology of the chain complex $P_* \otimes_{\mathbb{Z}G} M$. Using the 1-1 correspondence between $\mathbb{Z}G$-modules M and local coefficient systems $\mathcal{M}$ on an Eilenberg-MacLane space K, we get natural isomorphisms $H_*(K; \mathcal{M}) \simeq H_*(G; M)$ and $H^*(K; \mathcal{M}) \simeq H^*(G; M)$. Henceforth we will use the same notation to denote $\mathbb{Z}G$-modules and the corresponding local systems on $K(G, 1)$'s.

Group pairs. We now discuss relative (co)homology following [7]. Let G be a group, and $\mathcal{H} := \{H_i\}_{i \in I}$ an indexed collection of (not necessarily distinct) subgroups. We refer to $(G, \mathcal{H})$ as a *group pair.* Let $\amalg_i K(H_i, 1) \xrightarrow{f} K(G, 1)$ be the map induced by the inclusions $H_i \to G$, and let K be the mapping cylinder of f. We therefore have a pair of spaces $(K, \amalg_i K(H_i, 1))$ since the domain of a map naturally embeds in the mapping cylinder. Given any $\mathbb{Z}G$-module M, we define the relative cohomology $H^*(G, \mathcal{H}; M)$

(respectively homology $H_*(G,\mathcal{H};M)$) to be the cohomology (resp. homology) of the pair $(K, \amalg_i K(H_i,1))$ with coefficients in the local system M. As in the absolute case, one can compute relative (co)homology groups using projective resolutions, see [7]. For each $i \in I$, let

$$\ldots \to Q_n(i) \to \ldots \to Q_0(i) \to \mathbb{Z} \to 0$$

be a resolution of $\mathbb{Z}$ by projective $\mathbb{Z}H_i$-modules, and let

$$\ldots \to P_n \to \ldots \to P_0 \to \mathbb{Z} \to 0$$

be a resolution of $\mathbb{Z}$ by projective $\mathbb{Z}G$-modules. The inclusions $H_i \to G$ induce $\mathbb{Z}H_i$-chain mappings $f_i : Q_*(i) \to P_*$, unique up to chain homotopy. We define a $\mathbb{Z}G$-chain complex Q_* to be $\oplus_i(\mathbb{Z}G \otimes_{\mathbb{Z}H_i} Q_*(i))$ with an augmentation

$$Q_0 \to \oplus_i(\mathbb{Z}G \otimes_{\mathbb{Z}H_i} \mathbb{Z})$$

induced by the augmentations $Q_0(i) \to \mathbb{Z}$; the chain mappings f_i yield a $\mathbb{Z}G$-chain mapping $f : Q_* \to P_*$. We let C_* be the algebraic mapping cylinder of f: this is the chain complex with $C_i := P_i \oplus Q_{i-1} \oplus Q_i$ with the boundary homomorphism given by

$$\partial(p_i, q_{i-1}, q_i) = (\partial p_i + f(q_{i-1}), -\partial q_{i-1}, \partial q_i + q_{i-1}). \tag{3.1}$$

We note that each C_i is clearly projective, a copy D_* of Q_* naturally sits in C_* as the third summand, and the quotient C_*/D_* is a chain complex of projective $\mathbb{Z}G$-modules. Proposition 1.2 of [7] implies that the relative homology (resp. cohomology) of the group pair $(G,\mathcal{H})$ with coefficients in a $\mathbb{Z}G$-module M (defined as above using local systems on Eilenberg-MacLane spaces) is canonically isomorphic to homology of the chain complex $(C_*/D_*) \otimes_{\mathbb{Z}G} M$ (resp. $Hom_{\mathbb{Z}G}((C_*/D_*), M)$).

Finiteness properties of groups. The (cohomological) dimension $dim(G)$ of a group G is n if n is the minimal integer such that there exists a resolution of $\mathbb{Z}$ by projective $\mathbb{Z}G$-modules:

$$0 \to P_n \to \ldots \to P_0 \to \mathbb{Z} \to 0.$$

Recall that G has cohomological dimension n if and only if n is the minimal integer so that $H^k(G,M) = 0$ for all $k > n$ and all $\mathbb{Z}G$-modules M. Moreover, if $dim(G) < \infty$ then

$$dim(G) = \sup\{n \mid H^n(G,F) \neq 0 \text{ for some free } \mathbb{Z}G\text{-module } F\},$$

see [12, Ch. VIII, Proposition 2.3]. If

$$1 \to G_1 \to G \to G_2 \to 1$$

is a short exact sequence then $dim(G) \leq dim(G_1) + dim(G_2)$, [12, Ch. VIII, Proposition 2.4]. If $G' \subset G$ is a subgroup then $dim(G') \leq dim(G)$.

A *partial resolution* of a $\mathbb{Z}G$-module A is an exact sequence $\mathbb{Z}G$-modules:

$$P_n \to \ldots \to P_0 \to A \to 0.$$

If A_*:

$$... \to A_n \to A_{n-1} \to \ldots \to A_0 \to A \to 0$$

is a chain complex then we let $[A_*]_n$ denote the *n-truncation of* A_*, i.e.

$$A_n \to \ldots \to A_0 \to A \to 0.$$

A group G is of type FP_n if there exists a partial resolution of $\mathbb{Z}$ by finitely generated projective $\mathbb{Z}G$-modules:

$$P_n \to ... \to P_0 \to \mathbb{Z} \to 0.$$

The group G is of type FP (resp. FL) if there exists a finite resolution of $\mathbb{Z}$ by finitely generated projective (resp. free) $\mathbb{Z}G$-modules. We will also refer to groups of type FP as *groups of finite type.*

Lemma 3.2. *1. If G is of type FP then $dim(G) = n$ if and only if*

$$n = \max\{i : H^i(G, \mathbb{Z}G) \neq 0\}.$$

2. If $dim(G) = n$ and G is of type FP_n then there exists a resolution of $\mathbb{Z}$ by finitely generated projective $\mathbb{Z}G$-modules:

$$0 \to P_n \to ... \to P_0 \to \mathbb{Z} \to 0.$$

In particular G is of type FP.

Proof. The first assertion follows from [12, Ch. VIII, Proposition 5.2]. We prove 2. Start with a partial resolution

$$P_n \to P_{n-1} \to ... \to P_0 \to \mathbb{Z} \to 0$$

where each P_i is finitely generated projective. By [12, Ch. VIII, Lemma 2.1], the kernel $Q_n := \ker[P_{n-1} \to P_{n-2}]$ is projective. However P_n maps onto Q_n, hence Q_n is also finitely generated. Thus replacing P_n with Q_n we get the required resolution. ☐

Examples of groups of type FP and FL are given by fundamental groups of finite Eilenberg-MacLane complexes, or more generally, groups acting freely cocompactly on acyclic complexes. According to the Eilenberg-Ganea theorem, if G is a finitely presentable group of type FL then G admits a finite $K(G, 1)$ of dimension $\max(dim(G), 3)$.

Let G be a group, let $\mathcal{H} := \{H_i\}_{i \in I}$ be an indexed collection of subgroups, and let

$$\epsilon : \oplus_i (\mathbb{Z}G \otimes_{\mathbb{Z}H_i} \mathbb{Z}) \to \mathbb{Z}$$

be induced by the usual augmentation $\mathbb{Z}G \to \mathbb{Z}$. Then the group pair $(G, \mathcal{H})$ *has finite type* if the $\mathbb{Z}G$-module $Ker(\epsilon)$ admits a finite length resolution by finitely generated projective $\mathbb{Z}G$-modules. If the index set I is finite and the groups G and H_i are of type FP then the group pair $(G, \mathcal{H})$ is of finite type, and one obtains the desired resolution of $Ker(\epsilon)$ using the quotient C_*/D_* where (C_*, D_*) is the pair given by the algebraic mapping cylinder construction (3.1).

For the next three topics, the reader may consult [5, 6, 7, 12, 14].

Duality groups. Let G be a group of type FP. Then G is an *n-dimensional duality group* if $H^i(G;\mathbb{Z}G) = \{0\}$ when $i \neq n = dim(G)$, and $H^i(G;\mathbb{Z}G)$ is torsion-free, [6]. There is an alternate definition of duality groups involving isomorphisms $H^i(G;M) \simeq H_{n-i}(G;D\otimes M)$ for a suitable dualizing module D and arbitrary $\mathbb{Z}G$-modules M, see [6, 12]. Examples of duality groups include:

1. The fundamental groups of compact aspherical manifolds with aspherical boundary, where the inclusion of each boundary component induces a monomorphism of fundamental groups.

2. Torsion-free S-arithmetic groups, [6, 9].

3. 2-dimensional one-ended groups of type FP_2 [5, Proposition 9.17]; for instance torsion-free, one-ended, one-relator groups.

4. Any group which can act freely, cocompactly, and simplicially on an acyclic simplicial complex X, where $H^i_c(X)$ vanishes except in dimension n, and $H^n_c(X)$ is torsion-free.

Poincaré duality groups. These form a special class of duality groups. If G is an n-dimensional duality group and $H^n(G;\mathbb{Z}G) = \mathbb{Z}$, then G is an *n-dimensional Poincare duality group* ($PD(n)$ group). As in the case of duality groups, there is an alternate definition involving isomorphisms $H^i(G;M) \simeq H_{n-i}(G;D\otimes M)$ where M is an arbitrary $\mathbb{Z}G$-module and the orientation $\mathbb{Z}G$-module D is isomorphic to $\mathbb{Z}$ as an abelian group. Examples include:

1. Fundamental groups of closed aspherical manifolds.

2. Fundamental groups of aspherical finite Poincare complexes. Recall that an (orientable) *Poincare complex of formal dimension n* is a finitely dominated complex K together with a fundamental class $[K] \in H_n(K;\mathbb{Z})$ so that the cap product operation $[K]\cap : H^k(K;M) \to H_{n-k}(K;M)$ is an isomorphism for every local system M on K and for $k = 0, \dots, n$.

3. Any group which can act freely, cocompactly, and simplicially on an acyclic simplicial complex X, where X has the same compactly supported cohomology as $\mathbb{R}^n$.

4. Each torsion-free Gromov-hyperbolic group G whose boundary is a homology manifold with the homology of sphere (over $\mathbb{Z}$), see [4]. Note that every such group is the fundamental group of a finite aspherical Poincare complex, namely the G-quotient of a Rips complex of G.

Below are several useful facts about Poincare duality groups (see [12]):

(a) If G is a $PD(n)$ group and $G' \subset G$ is a subgroup then G' is a $PD(n)$ group if and only if the index $[G : G']$ is finite.

(b) If G is a $PD(n)$ group which is contained in a torsion-free group G' as a finite index subgroup, then G' a $PD(n)$ group.

(c) If $G \times H$ is a $PD(m)$ group then G and H are $PD(n)$ and $PD(k)$ groups, where $m = n + k$.

(d) If $G \rtimes H$ is a semi-direct product where G is a $PD(n)$-group and H is a $PD(k)$-group, then $G \rtimes H$ is a $PD(n + k)$-group. See [6, Theorem 3.5].

There are several questions about $PD(n)$ groups and their relation with fundamental groups of aspherical manifolds. It was an open question going back to Wall

[35] whether every $PD(n)$ group is the fundamental group of a closed aspherical manifold. The answer to this is yes in dimensions 1 and 2, [33, 16, 17]. Recently, Davis in [13] gave examples for $n \geq 4$ of $PD(n)$ groups which do not admit a finite presention, and these groups are clearly not fundamental groups of compact manifolds. This leaves open several questions:

1. Is every finitely presented $PD(n)$ group the fundamental group of a compact aspherical manifold?

2. A weaker version of 1: Is every finitely presented $PD(n)$ group the fundamental group of a finite aspherical complex? Equivalently, by Eilenberg-Ganea, one may ask if every such group is of type FL.

3. Does every $PD(n)$ group act freely and cocompactly on an acyclic complex? We believe this question is open for groups of type FP. One can also ask if every $PD(n)$ group acts freely and cocompactly on an acyclic n-manifold.

Poincare duality pairs. Let G be an $(n-1)$-dimensional group of type FP, and let $H_1, \dots, H_k \subset G$ be $PD(n-1)$ subgroups of G. Then the group pair $(G; H_1, \dots, H_k)$ is an *n-dimensional Poincare duality pair, or $PD(n)$ pair*, if the double of G over the H_i's is a $PD(n)$ group. We recall that the double of G over the H_i's is the fundamental group of the graph of groups $\mathcal{G}$, where $\mathcal{G}$ has two vertices labelled by G, k edges with the i^{th} edge labelled by H_i, and edge monomorphisms are the inclusions $H_i \to G$. An alternate homological definition of $PD(n)$ pairs is the following: a group pair $(G, \{H_i\}_{i\in I})$ is a $PD(n)$ pair if it has finite type, and $H^*(G, \{H_i\}; \mathbb{Z}G) \simeq H^*_c(\mathbb{R}^n)$. For a discussion of these and other equivalent definitions, see [7, 14]. We will sometimes refer to the system of subgroups $\{H_i\}$ as the *peripheral structure* of the $PD(n)$ pair, and the H_i's as peripheral subgroups. The first class of examples of duality groups mentioned above have natural peripheral structure which makes them $PD(n)$ pairs. In [27] we proved that if G is a torsion-free Gromov-hyperbolic group whose boundary is homeomorphic to the Sierpinski carpet S, then $(G; H_1, ..., H_k)$ is a $PD(3)$ group pair, where H_i's are representatives of conjugacy classes of stabilizers of the peripheral circles of S in $\partial_\infty G$. If $(G; H_1, \dots, H_k)$ is a $PD(n)$ pair, where G and each H_i admit a finite Eilenberg-MacLane space X and Y_i respectively, then the inclusions $H_i \to G$ induce a map $\sqcup_i Y_i \to X$ (well-defined up to homotopy) whose mapping cylinder C gives a *Poincare pair* $(C; \sqcup_i Y_i)$, i.e. a pair which satisfies Poincare duality for manifolds with boundary with local coefficients (where $\sqcup_i Y_i$ serves as the boundary of C). Conversely, if (X, Y) is a Poincare pair where X is aspherical and Y is a union of aspherical components Y_i, then $(\pi_1(X); \pi_1(Y_1), \dots, \pi_1(Y_k))$ is a $PD(n)$ pair.

Lemma 3.3. *Let $(G, \{H_i\})$ be a $PD(n)$ pair, where G is not a $PD(n-1)$ group. Then the subgroups H_i are pairwise non-conjugate maximal $PD(n-1)$ subgroups.*

Proof. If H_i is conjugate to H_j for some $i \neq j$, then the double $\hat{G}$ of G over the peripheral subgroups would contain an infinite index subgroup isomorphic to the $PD(n)$ group $H_i \times \mathbb{Z}$. The group $\hat{G}$ is a $PD(n)$ group, which contradicts property (a) of Poincare duality groups listed above.

We now prove that each H_i is maximal. Suppose that $H_i \subset H \subset G$, where $H \neq H_i$ is a $PD(n-1)$ group. Then $[H : H_i] < \infty$. Pick $h \in H - H_i$. Then there exists a finite index subgroup $F_i \subset H_i$ which is normalized by h. Consider the double $\hat{G}$ of G along the collection of subgroups $\{H_i\}$, and let $\hat{G} \curvearrowright T$ be the associated action on the Bass-Serre tree. Since G is not a $PD(n-1)$ group, $H_i \neq G$ for each i, and so there is a unique vertex $v \in T$ fixed by G. The involution of the graph of groups defining $\hat{G}$ induces an involution of $\hat{G}$ which is unique up to an inner automorphism; let $\tau : \hat{G} \to \hat{G}$ be an induced involution which fixes H_i elementwise. Then $G' := \tau(G)$ fixes a vertex v' adjacent to v, where the edge $\overline{vv'}$ is fixed by H_i. So $h' := \tau(h)$ belongs to $\tau(G) = G'$ but h' does not fix $\overline{vv'}$. Therefore the fixed point sets of h and h' are disjoint, which implies that $g := hh'$ acts on T as a hyperbolic automorphism. Since $h' \in Normalizer(\tau(F_i)) = Normalizer(F_i)$, we get $g \in Normalizer(F_i)$. Hence the subgroup F generated by F_i and g is a semi-direct product $F = F_i \rtimes \langle g \rangle$, and $\langle g \rangle \simeq \mathbb{Z}$ since g is hyperbolic. The group F is a $PD(n)$ group (by property (d)) sitting as an infinite index subgroup of the $PD(n)$ group G, which contradicts property (a). $\square$

4. Algebraic preliminaries

In this section we introduce a notion of "morphism" between inverse systems. Approximate isomorphisms, which figure prominently in the remainder of the paper, are maps between inverse (or direct) systems which fail to be isomorphisms in a controlled way, and for many purposes are as easy to work with as isomorphisms.

Approximate morphisms between inverse and direct systems. Recall that a partially ordered set I is *directed* if for each $i, j \in I$ there exists $k \in I$ such that $k \geq i, j$. An inverse system of (abelian) groups indexed by a directed set I is a collection of abelian groups $\{A_i\}_{i \in I}$ and homomorphisms (*projections*) $p_i^j : A_i \to A_j$, $i \geq j$ so that

$$p_i^i = id \text{ and } p_j^k \circ p_i^j = p_i^k$$

for any $i \leq j \leq k$. (One may weaken these assumptions but they will suffice for our purposes.) We will often denote the inverse system by $(A_\bullet, p_\bullet)$ or $\{A_i\}_{i \in I}$. Recall that a subset $I' \subset I$ of a partially ordered set is *cofinal* if for every $i \in I$ there is an $i' \in I'$ so that $i' \geq i$.

Let $\{A_i\}_{i \in I}$ and $\{B_j\}_{j \in J}$ be two inverse systems of (abelian) groups indexed by I and J, with the projection maps $p_i^{i'} : A_i \to A_{i'}$ and $q_j^{j'} : B_j \to B_{j'}$. The directed sets appearing later in the paper will be order isomorphic to $\mathbb{Z}_+$ with the usual order.

Definition 4.1. Let α be an order preserving, partially defined, map from I to J. Then α is *cofinal* if it is defined on a subset of the form $\{i \in I \mid i \geq i_0\}$ for some $i_0 \in I$, and the image of every cofinal subset $I' \subset I$ is a cofinal subset $\alpha(I') \subset J$.

Definition 4.2. Let $\alpha : I \to J$ be a cofinal map. Suppose that $(\{A_i\}_{i \in I}, p_\bullet)$ and $(\{B_j\}_{j \in J}, q_\bullet)$ are inverse systems. Then a family of homomorphisms $f_i : A_i \to B_{\alpha(i)}$, $i \in I$, is an *α-morphism from $\{A_i\}_{i \in I}$ to $\{B_j\}_{j \in J}$* if

$$q_{\alpha(i)}^{\alpha(i')} \circ f_i = f_i \circ p_i^{i'} \tag{4.3}$$

whenever $i, i' \in I$ and $i \geq i'$. The *saturation* $\hat{f}_\bullet^\bullet$ of the α-morphism $f_\bullet$ is the collection of maps $\hat{f}_i^j : A_i \to B_j$ of the form

$$q_{\alpha(k)}^j \circ f_k \circ p_i^k \ .$$

In view of (4.3) this definition is consistent, and $\hat{f}_\bullet^\bullet$ is compatible with the projection maps of $A_\bullet$ and $B_\bullet$.

Suppose that $\{A_i\}_{i\in I}$, $\{B_j\}_{j\in J}$, $\{C_k\}_{k\in K}$ are inverse systems, $\alpha : I \to J$, $\beta : J \to K$ are cofinal maps. Then the composition of α- and β-morphisms

$$f_\bullet : A_\bullet \to B_\bullet, \quad g_\bullet : B_\bullet \to C_\bullet$$

is a γ-morphism for the cofinal map $\gamma = \beta \circ \alpha : I \to K$. (The composition $\beta \circ \alpha$ is defined on the subset $Domain(\alpha) \cap \alpha^{-1}(Domain(\beta))$ which contains $\{i : i \geq i_1\}$ where i_1 is an upper bound for non-cofinal subset $\alpha^{-1}(J - Domain(\beta))$ in I.)

Definition 4.4. Let $A_\bullet \stackrel{f_\bullet}{\to} B_\bullet$ be an α-morphism of inverse systems $(A_\bullet, p_\bullet), (B_\bullet, q_\bullet)$.

1. When I is totally ordered, we define $Im(\hat{f}^j_\bullet)$, the *image* of $f_\bullet$ in B_j, to be $\cup\{Im(\hat{f}^j_i : A_i \to B_j) \mid \alpha(i) \geq j\}$.

2. Let $\omega : I \to I$ be a function. Then $f_\bullet$ is an ω-*approximate monomorphism* if for every $i \in I$ we have

$$Ker(A_{\omega(i)} \stackrel{f_{\omega(i)}}{\longrightarrow} B_{\alpha(\omega(i))}) \subset Ker(A_{\omega(i)} \stackrel{p_\bullet}{\longrightarrow} A_i).$$

3. Suppose I is totally ordered. If $\bar{\omega} : J \to J$ is a function, then $f_\bullet$ is an $\bar{\omega}$-*approximate epimorphism* if for every $j \in J$ we have:

$$Im(B_{\bar{\omega}(j)} \stackrel{q_\bullet}{\longrightarrow} B_j) \subset Im(\hat{f}^j_\bullet).$$

4. Suppose I is totally ordered. If $\omega : I \to I$ and $\bar{\omega} : J \to J$ are functions, then f is an $(\omega, \bar{\omega})$-*approximate isomorphism* if both 2 and 3 hold.

We will frequently suppress the functions α, ω, $\bar{\omega}$ when speaking of morphisms, approximate monomorphisms (epimorphisms, isomorphisms). Note that the inverse limit of an approximate monomorphism (epimorphism, isomorphism) is a monomorphism (epimorphism, isomorphism) of inverse limits.

Note that an α-morphism induces a homomorphism between inverse limits, since for each cofinal subset $J' \subset J$ we have:

$$\lim_{\substack{\longleftarrow \\ j\in J}} B_j \cong \lim_{\substack{\longleftarrow \\ j\in J'}} B_j \ .$$

Similarly, an approximate isomorphism of inverse systems induces an isomorphism of their inverse limits. However the converse is not true. For instance, let $A_i := \mathbb{Z}$ for each $i \in \mathbb{N}$, where $\mathbb{N}$ has the usual order. Let

$$p_i^{i-n} : A_i \to A_{i-n} \ \text{ be the index } n \text{ inclusion.}$$

It is clear that the inverse limit of this system is zero. We leave it to the reader to verify that the system $(A_\bullet, p_\bullet)$ is not approximately isomorphic to zero inverse system.

We have similar definitions for homomorphisms of direct systems. A direct system of (abelian) groups indexed by a directed set I is a collection of abelian groups $\{A_i\}_{i\in I}$ and homomorphisms (*projections*) $p_i^j : A_i \to A_j$, $i \leq j$ so that

$$p_i^i = id, \quad p_j^k \circ p_i^j = p_i^k$$

for any $i \leq j \leq k$. We often denote the direct system by $(A_\bullet, p_\bullet)$. Let $\{A_i\}_{i\in I}$ and $\{B_j\}_{j\in J}$ be two direct systems of (abelian) groups indexed by directed sets I and J, with projection maps $p_i^{i'} : A_i \to A_{i'}$ and $q_j^{j'} : B_j \to B_{j'}$.

Definition 4.5. Let $\alpha : I \to J$ be a cofinal map. Then a family of homomorphisms $f_i : A_i \to B_{\alpha(i)}$, $i \in I$, is a *α-morphism of the direct systems $\{A_i\}_{i\in I}$ and $\{B_j\}_{j\in J}$* if

$$q_{\alpha(i)}^{\alpha(i')} \circ f_i = f_{i'} \circ p_i^{i'}$$

whenever $i \le i'$. We define the saturation $\hat{f}_\bullet^\bullet$ the same way as for morphisms of inverse systems.

Definition 4.6. Let $f_\bullet : A_\bullet \to B_\bullet$ be an α-morphism of direct systems:

$$f_\bullet = \{f_i : A_i \to B_{\alpha(i)}, i \in I\}.$$

1. When I is totally ordered we define $Im(\hat{f}_\bullet^j)$, the *image* of $f_\bullet$ in B_j, to be $\cup\{Im(\hat{f}_i^j) \mid \alpha(i) \le j\}$.

2. Let $\omega : I \to I$ be a function. Then $f_\bullet$ is an *ω-approximate monomorphism* if for every $i \in I$ we have

$$Ker(A_i \xrightarrow{f_i} B_{\alpha(i)}) \subset Ker(A_i \xrightarrow{p_\bullet} A_{\omega(i)}).$$

3. Suppose I is totally ordered, and $\bar{\omega} : J \to J$ is a function. $f_\bullet$ is an *$\bar{\omega}$-approximate epimorphism* if for every $j \in J$ we have:

$$Im(B_j \xrightarrow{q_\bullet} B_{\bar{\omega}(j)}) \subset Im(\hat{f}_\bullet^{\bar{\omega}(j)}).$$

4. Suppose I is totally ordered and $\omega : I \to I$ and $\bar{\omega} : J \to J$ are functions. Then f is an *$(\omega, \bar{\omega})$-approximate isomorphism* if both 2 and 3 hold.

An inverse (direct) system $A_\bullet$ is said to be constant if $A_i = A_j$ and $p_j^i = id$ for each i, j. An inverse (direct) system $A_\bullet$ is *approximately constant* if there is an approximate isomorphism between it and a constant system (in either direction). Likewise, an inverse or direct system is *approximately zero* if it is approximately isomorphic to a zero system.

The proof of the following lemma is straightforward and is left to the reader.

Lemma 4.7. *The composition of two approximate monomorphisms (epimorphisms, isomorphisms) is an approximate monomorphism (epimorphism, isomorphism).*

The remaining material in this section relates to the category theoretic behavior of approximate morphisms and a comparison with pro-morphisms, and it will not be used elsewhere in the paper.

In what follows $(A_\bullet, p_\bullet)$ and $(B_\bullet, q_\bullet)$ will once again denote inverse systems indexed by I and J respectively. However, for simplicity we will assume that I and J are both totally ordered.

Definition 4.8. Let $f_\bullet : A_\bullet \to B_\bullet$ be an α-morphism with saturation $\hat{f}_\bullet^\bullet$. The *kernel* of $f_\bullet$ is the inverse system $\{K_i\}_{i\in I}$ where $K_i := Ker(f_i : A_i \to B_{\alpha(i)})$ with the projection maps obtained from the projections of $A_\bullet$ by restriction. We define the *image* of $f_\bullet$ to be the inverse system $\{D_j\}_{j\in J}$ where $D_j := Im(\hat{f}_\bullet^j)$, with the projections coming from the projections of $B_\bullet$. Note that D_j is a subgroup of B_j, $j \in J$. We also define the *cokernel $coKer(f_\bullet)$ of $f_\bullet$*, as the inverse system $\{C_j\}_{j\in J}$ where $C_j := B_j/D_j$.

An inverse (respectively direct) system of abelian groups $A_\bullet$ is *pro-zero* if for every $i \in I$ there exists $j \geq i$ such that $p^i_j : A_j \to A_i$ (resp. $p^j_i : A_i \to A_j$) is zero (see [1, Appendix 3]). Using this language we may reformulate the definitions of approximate monomorphisms:

Lemma 4.9. *Let $f_\bullet : A_\bullet \to B_\bullet$ be a morphism of inverse systems of abelian groups. Then*

1. $f_\bullet$ is an approximate monomorphism iff its kernel $K_\bullet := Ker(f_\bullet)$ is pro-zero.

2. $f_\bullet$ is an approximate epimorphism iff its cokernel is a pro-zero inverse system.

3. $f_\bullet$ is an approximate isomorphism iff both $Ker(f_\bullet)$ and $coKer(f_\bullet)$ are pro-zero systems.

Proof. This is immediate from the definitions. $\square$

For a fixed cofinal map $\alpha : I \to J$, the collection of α-morphisms from $A_\bullet$ to $B_\bullet$ forms an abelian group the obvious way. In order to compare morphisms $A_\bullet \to B_\bullet$ with different index maps $I \to J$, we introduce an equivalence relation:

Definition 4.10. Let $f : A_\bullet \to B_\bullet$ and $g : A_\bullet \to B_\bullet$ be morphisms with saturations $\hat{f}^\bullet_\bullet$ and $\hat{g}^\bullet_\bullet$. Then $f_\bullet$ *is equivalent* $g_\bullet$ if there is a cofinal function $\rho : J \to I$ so that for all $j \in J$, both $\hat{f}^j_{\rho(j)}$ and $\hat{g}^j_{\rho(j)}$ are defined, and they coincide.

This equivalence relation is compatible with composition of approximate morphisms. Hence we obtain a category *Approx* where the objects are inverse systems of abelian groups and the morphisms are equivalence classes of approximate morphisms. An *approximate inverse* for an approximate morphism $f_\bullet$ is an approximate morphism $g_\bullet$ which inverts $f_\bullet$ in *Approx*.

Lemma 4.11. *Suppose $I, J \cong \mathbb{Z}_+$, $D_\bullet$ is a sub inverse system of $A_\bullet$ (i.e. $D_i \subset A_i$, $i \in I$), and let $Q_\bullet$ be the quotient system: $Q_i := A_i/D_i$. Then*

1. The morphism $A_\bullet \to Q_\bullet$ induced by the canonical epimorphisms $A_i \to Q_i$ has an approximate inverse iff $D_\bullet$ is a pro-zero system.

2. The morphism $D_\bullet \to A_\bullet$ defined by the inclusion homomorphisms $D_i \to A_i$ has an approximate inverse iff $Q_\bullet$ is a pro-zero system.

3. If $f_\bullet : A_\bullet \to B_\bullet$ is a morphism, $Ker(f_\bullet)$ is zero (i.e. $Ker(f_\bullet)_i = \{0\}$ for all $i \in I$), and $Im(f_\bullet) = B_\bullet$, then $f_\bullet$ has an approximate inverse.

Proof. We leave the "only if" parts of 1 and 2 to the reader.

When $D_\bullet$ is pro-zero the map $\beta : I \to I$ defined by

$$\beta(i) := \max\{i' \mid D_i \subset Ker(A_i \to A_{i'})\}$$

is cofinal. Let $g_\bullet : Q_\bullet \to A_\bullet$ be the β-morphism where $g_i : A_i/D_i = Q_i \to A_{\beta(i)}$ is induced by the projection $A_i \to A_{\beta(i)}$. One checks that $g_\bullet$ is an approximate inverse for $A_\bullet \to Q_\bullet$.

Suppose $Q_\bullet$ is pro-zero. Define a cofinal map $\beta : I \to I$ by

$$\beta(i) := \max\{i' \mid Im(A_i \to A_{i'}) \subset D_{i'}\},$$

and let $g_\bullet : A_\bullet \to D_\bullet$ be the β-morphism where $g_i : A_i \to D_{\beta(i)}$ is induced by the projection $A_i \to A_{\beta(i)}$. Then $g_\bullet$ is an approximate inverse for the inclusion $D_\bullet \to A_\bullet$.

Now suppose $f_\bullet : A_\bullet \to B_\bullet$ is an α-morphism with zero kernel and cokernel. Let $J' := \alpha(I) \subset J$, and define $\beta' : J' \to I$ by $\beta'(j) = \min \alpha^{-1}(j)$. Define a cofinal map $\sigma : J \to J'$ by $\sigma(j) := \max\{j' \in J' \mid j' \leq j\}$; let $\beta : J \to I$ be the composition $\beta' \circ \sigma$, and define a β-morphism $g_\bullet$ by $g_j := f_{\beta(j)}^{-1} \circ q_j^{\sigma(j)}$. Then $g_\bullet$ is the desired approximate inverse for $f_\bullet$. ☐

Lemma 4.12. *Let $f_\bullet : A_\bullet \to B_\bullet$ be a morphism.*

1. If $f_\bullet$ has an approximate inverse then it is an approximate isomorphism.

2. If $f_\bullet$ is an approximate isomorphism and $I, J \cong \mathbb{Z}_+$ then $f_\bullet$ has an approximate inverse.

Proof. Let $f_\bullet : A_\bullet \to B_\bullet$ and $g_\bullet : B_\bullet \to A_\bullet$ be α and β morphisms respectively, and let $g_\bullet$ be an approximate inverse for $f_\bullet$. Since $h_\bullet := g_\bullet \circ f_\bullet$ is equivalent to $id_{A_\bullet}$ then for all i there is an $i' \geq i$ so that $\hat{h}_{i'}^i$ is defined and $\hat{h}_{i'}^i = p_{i'}^i$. Letting $\gamma := \beta \circ \alpha$ we have, by the definition of the saturation $\hat{h}_\bullet^\bullet$, $p_{i'}^i = \hat{h}_{i'}^i = p_{\gamma(i)}^i \circ h_{i'}$. So $Ker(h_{i'}) \subset Ker(p_{i'}^i)$. Thus $f_\bullet$ is an approximate monomorphism. The proof that $f_\bullet$ is an approximate epimorphism is similar.

We now prove part 2. Let $\{K_i\}_{i \in I}$ be the kernel of $f_\bullet$, let $\{Q_i\}_{i \in I} = \{A_i/K_i\}_{i \in I}$ be the quotient system, and let $\{D_j\}_{j \in J}$ be the image of $f_\bullet$. Then $f_\bullet$ may be factored as $f_\bullet = t_\bullet \circ s_\bullet \circ r_\bullet$ where $r_\bullet : A_\bullet \to Q_\bullet$ is induced by the epimorphisms $A_i \to A_i/K_i$, $s_\bullet : Q_\bullet \to D_\bullet$ is induced by the homomorphisms of quotients, and $t_\bullet : D_\bullet \to B_\bullet$ is the inclusion. By Lemma 4.11, $s_\bullet$ has an approximate inverse. When the kernel and cokernel of $f_\bullet$ are pro-zero then $r_\bullet$ and $t_\bullet$ also admit approximate inverses by Lemma 4.11. Hence $f_\bullet$ has an approximate inverse in this case. ☐

Relation with Grothendieck's pro-morphisms. Below we relate the notions of α-morphisms, approximate monomorphisms (epimorphisms, isomorphisms) with Grothendieck's pro-morphisms. Strictly speaking this is unnecessary for the purposes of this paper, however it puts our definitions into perspective. Also, readers who prefer the language of pro-categories may use Lemma 4.14 and Corollary 4.15 to translate the theorems of sections 6 and 7 into pro-theorems.

Definition 4.13. Let $\{A_i\}_{i \in I}$, $\{B_j\}_{j \in J}$ be inverse systems. The group of pro-morphisms $proHom(A_\bullet, B_\bullet)$ is defined as

$$\varprojlim_{j \in J} \varinjlim_{i \in I} Hom(A_i, B_j)$$

(see [22], [1, Appendix 2], [15, Ch II, §1]). The *identity pro-morphism* is the element of $proHom(A_\bullet, A_\bullet)$ determined by $(id_{A_j})_{j \in I} \in \prod_j \varinjlim_{i \in I} Hom(A_i, A_j)$.

This yields a category[4] *Pro-Abelian* where the objects are inverses systems of abelian groups and the morphisms are the pro-morphisms. A *pro-isomorphism* is an isomorphism in this category.

[4]By relaxing the definition of inverse systems, this category becomes an abelian category, [1, Appendix 4]. However we will not discuss this further.

By the definitions of direct and inverse limits, an element of $proHom(A_\bullet, B_\bullet)$ can be represented by an admissible "sequence"

$$([h^j_{\rho(j)} : A_{\rho(j)} \to B_j])_{j\in J}$$

of equivalence classes of homomorphisms $h^j_{\rho(j)} : A_{\rho(j)} \to B_j$; here two homomorphisms $h^j_i : A_i \to B_j, h^j_k : A_k \to B_j$ are equivalent if there exists $\ell \geq i, k$ such that

$$h^j_i \circ p^i_\ell = h^j_k \circ p^k_\ell;$$

and the "sequence" is *admissible* if for each $j \geq j'$ there is an $i \geq \max\{\rho(j), \rho(j')\}$ so that

$$q^{j'}_j \circ h^j_{\rho(j)} \circ p^{\rho(j)}_i = h^{j'}_{\rho(j')} \circ p^{\rho(j')}_i.$$

Given a cofinal map $\alpha : I \to J$ between directed sets, we may construct[5] a function $\rho : J \to I$ so that $\alpha(\rho(j)) \geq j$ for all j; then any α-morphism $f_\bullet : A_\bullet \to B_\bullet$ induces an admissible sequence $([\hat{f}^j_{\rho(j)} : A_{\rho(j)} \to B_j]\}_{j\in J}$. The corresponding element $pro(f_\bullet) \in proHom(A_\bullet, B_\bullet)$ is independent of the choice of ρ by condition (4.3) of Definition 4.2.

Lemma 4.14. *1. If $f : A_\bullet \to B_\bullet$ and $g : A_\bullet \to B_\bullet$ are morphisms, then $pro(f) = pro(g)$ iff $f_\bullet$ is equivalent to $g_\bullet$. In other words, pro descends to a faithful functor from Approx to Pro-Abelian.*

2. When I, $J \cong \mathbb{Z}_+$ then every pro-morphism from $A_\bullet$ to $B_\bullet$ arises as $pro(f_\bullet)$ for some approximate morphism $f_\bullet : A_\bullet \to B_\bullet$. Thus pro descends to a fully faithful functor from Approx to Pro-Abelian in this case.

Proof. The first assertion follows readily from the definition of $proHom(A_\bullet, B_\bullet)$ and Definition 4.10.

Suppose I, $J \cong \mathbb{Z}_+$ and $\phi \in proHom(A_\bullet, B_\bullet)$ is represented by an admissible sequence

$$([h^j_{\rho_0(j)} : A_{\rho_0(j)} \to B_j])_{j\in J}.$$

We define $\rho : J \to I$ and another admissible sequence $(\bar{h}^j_{\rho(j)} : A_{\rho(j)} \to B_j)_{j\in J}$ representing ϕ by setting $\rho(0) = \rho_0(0)$, $\bar{h}^0_{\rho(0)} := h^0_{\rho_0(0)}$, and inductively choosing $\rho(j)$, $\bar{h}^j_{\rho(j)}$ so that $\rho(j) > \rho(j-1)$, $\bar{h}^j_{\rho(j)} := h^j_{\rho_0(j)} \circ p^{\rho_0(j)}_{\rho(j)}$ and $q^{j-1}_j \circ \bar{h}^j_{\rho(j)} = \bar{h}^{j-1}_{\rho(j-1)} \circ p^{\rho(j-1)}_{\rho(j)}$. Note that the mapping ρ is strictly increasing and hence cofinal. Now define a cofinal map $\alpha : \mathbb{Z}_+ \to \mathbb{Z}_+$ by setting $\alpha(i) := \max\{j \mid \rho(j) \leq i\}$ for $i \geq \rho(0) = \rho_0(0)$. We then get an α-morphism $f_\bullet : A_\bullet \to B_\bullet$ where $f_i := \bar{h}^{\alpha(i)}_{\rho(\alpha(i))} \circ p^{\rho(\alpha(i))}_i$. Clearly $pro(f_\bullet) = (\bar{h}^j_{\rho(j)})_{j\in J}$. □

Corollary 4.15. *Suppose $I, J \cong \mathbb{Z}_+$ and $f_\bullet : A_\bullet \to B_\bullet$ is a morphism. Then $f_\bullet$ is an approximate isomorphism iff $pro(f_\bullet)$ is a pro-isomorphism.*

Proof. By Lemma 4.12, $f_\bullet$ is an approximate isomorphism iff it represents an invertible element of *Approx*, and by Lemma 4.14 this is equivalent to saying that $pro(f_\bullet)$ is invertible in *Pro-Abelian*. □

[5]Using the axiom of choice we pick $\rho(j) \in \alpha^{-1}(j)$.

5. Recognizing groups of type FP_n

The main result in this section is Theorem 5.9, which gives a characterization of groups G of type FP_n in terms of nested families of G-chain complexes, and Lemma 5.1 which relates the cohomology of G with the corresponding cohomology of the G-chain complexes. A related characterization of groups of type FP_n appears in [11]. We will apply Theorem 5.9 and Lemma 5.1 in section 8.

Suppose for $i = 0, \dots, N$ we have an augmented chain complex $A_*(i)$ of projective $\mathbb{Z}G$-modules, and for $i = 1, \dots, N$ we have an augmentation preserving G-equivariant chain map $a_i : A_*(i-1) \to A_*(i)$ which induces zero on reduced homology in dimensions $< n$. Let G be a group of type FP_k, and let

$$0 \leftarrow \mathbb{Z} \leftarrow P_0 \leftarrow \dots \leftarrow P_k$$

be a partial resolution P_* of $\mathbb{Z}$ by finitely generated projective $\mathbb{Z}G$-modules. We assume that $k \le n \le N$.

Lemma 5.1. *Under the above conditions we have:*

1. There is an augmentation preserving G-equivariant chain mapping $P_* \to A_*(n)$.

2. If $k < n$ and $j_i : P_* \to A_*(0)$ are augmentation preserving G-equivariant chain mappings for $i = 1, 2$, then the compositions $P_* \xrightarrow{j_i} A_*(0) \to A_*(k)$ are G-equivariantly chain homotopic.

3. Suppose $k < n$ and $f : P_* \to A_*(0)$ is an augmentation preserving G-equivariant chain mapping. Then for any $\mathbb{Z}G$-module M, the map

$$H^i(f) : H^i(A_*(0); M) \to H^i(P_*; M)$$

carries the image $Im(H^i(A_*(n); M) \to H^i(A_*(0); M))$ isomorphically onto $H^i(P_*; M)$ for $i = 0, \dots k-1$. The map

$$H_i(f) : H_i(P_*; M) \to H_i(A_*(n); M)$$

is an isomorphism onto the image of $H_i(A_*(0); M) \to H_i(A_*(n); M)$ for $i = 0, \dots k-1$. The map

$$H_k(f) : H_k(P_*; M) \to H_k(A_*(n); M)$$

is onto the image of $H_k(A_*(0); M) \to H_k(A_*(n); M)$.

Proof of 1. We start with the diagram

$$\begin{array}{ccc} P_0 & & \\ \downarrow & & \\ \mathbb{Z} & \leftarrow & A_0(0). \end{array}$$

Then projectivity of P_0 implies that we can complete this to a commutative diagram by a $\mathbb{Z}G$-morphism $f_0 : P_0 \to A_0(0)$. Assume inductively that we have constructed a G-equivariant augmentation preserving chain mapping $f_j : [P_*]_j \to A_*(i)$. Then the image of the composition $P_{i+1} \xrightarrow{\partial} P_j \xrightarrow{f_j} A_j(j) \to A_j(j+1)$ is contained in the image of $A_{j+1}(j+1) \xrightarrow{\partial} A_j(j+1)$ since a_{j+1} induces zero on reduced homology.

So projectivity of P_{j+1} allows us to extend f_j to a G-equivariant chain mapping $f_{j+1} : [P_*]_{j+1} \to A_*(j+1)$.

Proof of 2. Similar to the proof of 1: use induction and projectivity of the P_ℓ's.

Proof of 3. Let $\rho_* : [A_*(n)]_k \to P_*$ be a G-equivariant chain mapping constructed using the fact that $H_i(P_*) = \{0\}$ for $i < k$. Consider the compositions

$$\alpha_{k-1} : [P_*]_{k-1} \xrightarrow{f_*} [A_*(0)]_{k-1} \to [A_*(n)]_{k-1} \xrightarrow{\rho_*} P_*$$

and

$$\beta_k : [A_*(0)]_k \to [A_*(n)]_k \xrightarrow{\rho_*} [P_*]_k \xrightarrow{f_*} [A_*(0)]_k \to A_*(n).$$

Both are (G-equivariantly) chain homotopic to the inclusions; the first one since P_* is a partial resolution, and the second by applying assertion 2 to the chain mapping $[A_*(0)]_k \to A_*(0)$. Assertion 3 follows immediately from this. □

We note that this lemma did not require any finiteness assumptions on the $\mathbb{Z}G$-modules $A_i(j)$. Suppose now that the group G satisfies the above assumptions and let $G \curvearrowright X$ be a free simplicial action on a uniformly $(n-1)$-acyclic locally finite metric simplicial complex X, $k \leq n-1$. Then by part 1 of the previous lemma we have a G-equivariant augmentation-preserving chain mapping $f : P_* \to C_*(X)$. Let $K \subset X$ be the support of the image of f. It is clear that K is G-invariant and K/G is compact. As a corollary of the proof of the previous lemma, we get:

Corollary 5.2. *Under the above assumptions the direct system of reduced homology groups* $\{\tilde{H}_i(N_R(K))\}_{R\geq 0}$ *is approximately zero for each* $i < k$.

Proof. Given $R > 0$ we consider the system of chain complexes $A_*(0) := C_*(N_R(K))$, $A_*(1) = A_*(2) = ... = A_*(N) = C_*(X)$. The mapping $[A_*(0)]_k \xrightarrow{\beta_k} A_*(N) = C_*(X)$ from the proof of Lemma 5.1 is chain homotopic to the inclusion via a G-equivariant homotopy h_R. On the other hand, this map factors through P_*, hence it induces zero mapping of the reduced homology groups

$$\tilde{H}_i(N_R(K)) \xrightarrow{0} \tilde{H}_i(Support(Im(\beta_k))), \ \ i < k.$$

The support of $Im(h_R)$ is contained in $N_{R'}(K)$ for some $R' < \infty$, since h_R is G-equivariant. Hence the inclusion $N_R(K) \to N_{R'}(K)$ induces zero map of $\tilde{H}_i(\cdot)$ for $i < k$. □

Before stating the next corollary, we recall the following fact:

Lemma 5.3. *(See [12].) Let* $G \curvearrowright X$ *be a discrete, free, cocompact action of a group on a simplicial complex. Then the complex of compactly supported simplicial cochains* $C^*_c(X)$ *is canonically isomorphic to the complex* $Hom_{\mathbb{Z}G}(C_*(X); \mathbb{Z}G)$*; in particular, the compactly supported cohomology of* X *is canonically isomorphic to* $H^*(X/G; \mathbb{Z}G)$.

In the next corollary we assume that G, P_*, X, f, K are as above, in particular, X is a uniformly $(n-1)$-acyclic locally finite metric simplicial complex, $k \leq n-1$, $P_k \to ... \to P_0 \to \mathbb{Z} \to 0$ is a resolution by finitely generated projective $\mathbb{Z}G$ modules.

Corollary 5.4. *1. For any local coefficient system (*$\mathbb{Z}G$*-module)* M *the family of maps*

$$H^i(N_R(K)/G; M) \xrightarrow{f^i_R} H^i(P_*; M)$$

defines a morphism between the inverse system $\{H^i(N_R(K)/G; M)\}_{R\geq 0}$ *and the constant inverse system* $\{H^i(P_*; M)\}_{R\geq 0}$ *which is an approximate isomorphism when* $0 \leq i < k$.

2. The map

$$H_c^i(N_R(K)) \simeq H^i(N_R(K)/G; \mathbb{Z}G) \xrightarrow{f_R^i} H^i(P_*; \mathbb{Z}G)$$

is an approximate isomorphism when $0 \leq i < k$.

3. The $\mathbb{Z}G$*-chain map*

$$f_{R,*} : P_* \to C_*(N_R(K))$$

induces a homomorphism of homology groups

$$f_{R,i} : \tilde{H}_i(P_*, \mathbb{Z}G) \to \tilde{H}_i(N_R(K))$$

which is an approximate isomorphism for $0 \leq i < k$.

Proof. 1. According to Corollary 5.2 the direct system of reduced homology groups $\{\tilde{H}_i(N_R(K))\}$ is pro-zero for each $i < k$. Thus for $N > k$ we have a sequence of integers $R_0 = 0 < R_1 < R_2 < ... < R_N$ so that the maps

$$\tilde{H}_i(N_{R_j}(K)) \to \tilde{H}_i(N_{R_{j+1}}(K))$$

are zero for each $j < N, i < k$. We now apply Lemma 5.1 where $A_*(j) := C_*(N_{R_j}(K))$.

2. This follows from part 1 and Lemma 5.3.

3. Note that $\tilde{H}_i(P_*, \mathbb{Z}G) \simeq \{0\}$ for $i < k$; this follows directly from the definition of a group of type FP_k. Thus the assertion follows from Corollary 5.2. □

There is also an analog of Corollary 5.4 which does not require a group action:

Lemma 5.5. *Let* X *and* Y *be bounded geometry metric simplicial complexes, where* Y *is uniformly* $(k-1)$*-acyclic and* X *is uniformly* k*-acyclic. Suppose* $C_*(Y) \xrightarrow{f} C_*(X)$ *is a uniformly proper chain mapping, and* $K := Support(Im(f)) \subset X$*. Then*

1. The induced map on cohomology

$$H_c^i(f) : H_c^i(N_R(K)) \to H_c^i(Y)$$

defines a morphism between the inverse system $\{H_c^i(N_R(K))\}_{R\geq 0}$ *and the constant inverse system* $\{H_c^i(Y)\}_{R\geq 0}$ *which is an approximate isomorphism for* $0 \leq i < k$*, and an approximate monomorphism for* $i = k$.

2. The approximate isomorphism approximately respects support in the following sense. There is a function $\zeta : \mathbb{N} \to \mathbb{N}$ *so that if* $i < k$*,* $S \subset Y$ *is a subcomplex,* $T := Support(f_*(C_*(S))) \subset X$ *is the corresponding subcomplex of* X*, and* $\alpha \in Im(H_c^i(Y, \overline{Y - S}) \to H_c^i(Y))$*, then* α *belongs to the image of the composition*

$$H_c^i(N_R(K), \overline{N_R(K) - N_{\zeta(R)}(T)}) \to H_c^i(N_R(K)) \xrightarrow{H_c^i(f)} H_c^i(Y).$$

3. The induced map

$$\tilde{H}_i(f) : \{0\} \simeq \tilde{H}_i(Y) \to \tilde{H}_i(N_R(K))$$

is an approximate isomorphism for $0 \leq i < k$.

Proof. Since f is uniformly proper, using the uniform $(k-1)$-acyclicity of Y and uniform k-acyclicity of X, we can construct a direct system $\{\rho_R\}$ of uniformly proper chain mappings between the truncated chain complexes

$$[0 \leftarrow C_0(N_R(K)) \leftarrow \ldots \leftarrow C_k(N_R(K))] \stackrel{\rho_R}{\to} [0 \leftarrow C_0(Y) \leftarrow \ldots \leftarrow C_k(Y)]$$

so that the compositions $f \circ \rho_R$ are chain homotopic to the inclusions

$$[0 \leftarrow C_0(N_R(K)) \leftarrow \ldots \leftarrow C_k(N_R(K))]$$
$$\to [0 \leftarrow C_0(N_{R'}(K)) \leftarrow \ldots \leftarrow C_k(N_{R'}(K))]$$

(for $R' = \omega(R)$) via chain homotopies of bounded support. Moreover the restriction of the composition $\rho_R \circ f$ to the $(k-1)$-truncated chain complexes is chain homotopic to the identity via a chain homotopy with bounded support.

We first prove that the morphism of inverse systems defined by

$$H^i_c(f) : H^i_c(N_R(K)) \to H^i_c(Y)$$

is an approximate monomorphism. Suppose

$$\alpha \in Ker(H^i_c(f) : H^i_c(N_{R'}(K)) \to H^i_c(Y))$$

where $R' = \omega(R)$. Then $H^i(f \circ \rho_{R'})(\alpha) = 0$. But the restriction of $H^i(f \circ \rho_{R'})(\alpha)$ to $N_R(K)$ is cohomologous to the restriction of α to $N_R(K)$.

Since the restriction of the composition $\rho_R \circ f$ to the $(k-1)$-truncated chain complex $[C_*(Y)]_{k-1}$ is chain homotopic to the identity, it follows that

$$H^i_c(f) : H^i_c(N_R(K)) \to H^i_c(Y)$$

is an epimorphism for $R \geq 0$ and $i < k$.

Part 2 of the lemma follows immediately from the uniform properness of ρ_R and the coarse Lipschitz property of the chain homotopies constructed above.

We omit the proof of part 3 as it is similar to that of part 2. □

Note that in the above discussion we used finiteness assumptions on the group G to make conclusions about (co)homology of families of G-invariant chain complexes. Our next goal is to use existence of a family of chain complexes $A_*(i)$ of *finitely generated* projective $\mathbb{Z}G$ modules as in Lemma 5.1 to establish finiteness properties of the group G (Theorem 5.9). We begin with a homotopy-theoretic analog of Theorem 5.9.

Proposition 5.6. *Let G be a group, and let $X(0) \stackrel{a_1}{\to} X(1) \stackrel{a_2}{\to} \ldots \stackrel{a_{n+1}}{\to} X(n+1)$ be a diagram of free, simplicial G-complexes where $X(i)/G$ is compact for $i = 0, \ldots n+1$. If the maps a_i are n-connected for each i, then there is an $(n+1)$-dimensional free, simplicial G-complex Y where Y/G is compact and Y is n-connected.*

Proof. We build Y inductively as follows. Start with $Y_0 = G$ where G acts on Y_0 by left translation, and let $j_0 : Y_0 \to X(0)$ be any G-equivariant simplicial map. Inductively apply Lemma 5.7 below to the composition $Y_i \stackrel{j_i}{\to} X(i) \to X(i+1)$ to obtain Y_{i+1} and a simplicial G-map $j_{i+1} : Y_{i+1} \to X(i+1)$. Set $Y := Y_{n+1}$. □

Lemma 5.7. *Let Z and A be locally finite simplicial complexes with free cocompact simplicial G-actions, where $dim(Z) = k$, and Z is $(k-1)$-connected. Let $j : Z \to A$, be a null-homotopic G-equivariant simplicial map. Then we may construct a k-connected simplicial G-complex Z' by attaching (equivariantly) finitely many G-orbits of simplicial* [6] *$(k+1)$-cells to Z, and a G-map $j' : Z' \to A$ extending j.*

Proof. By replacing A with the mapping cylinder of j, we may assume that Z is a subcomplex of A and j is the inclusion map. Let A_k denote the k-skeleton of A. Since Z is $(k-1)$-connected, after subdividing A_k if necessary, we may construct a G-equivariant simplicial retraction $r : A_k \to Z$. For every $(k+1)$-simplex c in A, we attach a simplicial $(k+1)$-cell c' to Z using the composition of the attaching map of c with the retraction r. It is clear that we may do this G-equivariantly, and there will be only finitely many G-orbits of $(k+1)$-cells attached. We denote the resulting simplicial complex by Z', and note that the inclusion $j : Z \to A$ clearly extends (after subdivision of Z') to an equivariant simplicial map $j' : Z' \to A$.

We now claim that Z' is k-connected. Since we built Z' from Z by attaching $(k+1)$-cells, it suffices to show that $\pi_k(Z) \to \pi_k(Z')$ is trivial. If $\sigma : S^k \to Z$ is a simplicial map for some triangulation of S^k, we get a simplicial null-homotopy $\tau : D^{k+1} \to A$ extending σ. Let D^{k+1}_k denote the k-skeleton of D^{k+1}. The composition $D^{k+1}_k \xrightarrow{\tau} A \xrightarrow{r} Z \to Z'$ extends over each $(k+1)$-simplex Δ of D^{k+1}, since $\tau\big|_{\Delta} : \Delta \to A$ is either an embedding, in which case $r \circ \tau\big|_{\partial\Delta} : \partial\Delta \to Z'$ is null homotopic by the construction of Z', or $\tau\big|_{\Delta} : \Delta \to A$ has image contained in a k-simplex of A, and the composition

$$\partial\Delta \xrightarrow{\tau} A \xrightarrow{r} Z$$

is already null-homotopic. Hence the composition $S^k \xrightarrow{\sigma} Z \hookrightarrow Z'$ is null-homotopic. □

The next lemma is a homological analog of Lemma 5.7 which provides the inductive step in the proof of Theorem 5.9.

Lemma 5.8. *Let G be a group. Suppose $0 \leftarrow \mathbb{Z} \xleftarrow{\epsilon} P_0 \leftarrow \ldots \leftarrow P_k$ is a partial resolution by finitely generated projective $\mathbb{Z}G$-modules, and $\mathbb{Z} \xleftarrow{\epsilon} A_0 \leftarrow \ldots \leftarrow A_{k+1}$ is an augmented chain complex of finitely generated projective $\mathbb{Z}G$-modules. Let $j : P_* \to A_*$ be an augmentation preserving chain mapping which induces zero on homology groups*[7]*. Then we may extend P_* to a partial resolution P'_*:*

$$0 \leftarrow \mathbb{Z} \xleftarrow{\epsilon} P_0 \leftarrow \ldots \leftarrow P_k \leftarrow P_{k+1}$$

where P_{k+1} is finitely generated free, and j extends to a chain mapping $j' : P'_* \to A_*$.

Proof. By replacing A_* with the algebraic mapping cylinder of j, we may assume that P_* is embedded as a subcomplex of A_*, j is the inclusion, and for $i = 0, \ldots, k$, the chain group A_k splits as a direct sum of $\mathbb{Z}G$-modules $A_i = P_i \oplus Q_i$ where Q_i is finitely generated and projective. Applying the projectivity of Q_i, we construct a chain retraction from the k-truncation $[A_*]_k$ of A_* to P_*. Choose a finite set of generators $a_1, \ldots, a_\ell$ for the $\mathbb{Z}G$-module A_{k+1}. We let P_{k+1} be the free module of

[6] A simplicial cell is a simplicial complex PL-homeomorphic to a single simplex.
[7] We declare that $H_k(P_*) := Z_k(P_*)$.

rank ℓ, with basis $a'_1, \ldots, a'_\ell$, and define the boundary operator $\partial : P_{k+1} \to P_k$ by the formula $\partial(a'_i) = r(\partial(a_i))$. To see that $H_k(P'_*) = 0$, pick a k-cycle $\sigma \in Z_k(P_*)$. We have $\sigma = \partial\tau$ for some $\tau = \sum c_i a_i \in A_{k+1}$. Then $\sigma = r(\partial\tau) = \sum c_i r(\partial a_i) = \sum c_i \partial a'_i$; so σ is null-homologous in P'_*. The extension mapping $j' : P'_* \to A_*$ is defined by $a'_i \mapsto a_i, 1 \leq i \leq \ell$. □

Theorem 5.9. *Suppose for $i = 0, \ldots, N$ we have an augmented chain complex $A_*(i)$ of finitely generated projective $\mathbb{Z}G$-modules, and for $i = 1, \ldots, N$ we have an augmentation preserving G-equivariant chain map $a_i : A_*(i-1) \to A_*(i)$ which induces zero on reduced homology in dimensions $\leq n \leq N$.*

Then there is a partial resolution

$$0 \leftarrow \mathbb{Z} \leftarrow F_0 \leftarrow \ldots \leftarrow F_n$$

of finitely generated free $\mathbb{Z}G$-modules, and a G-equivariant chain mapping $f : F_* \to A(n)$. In particular, G is a group of type FP_n.

Proof. Define F_0 to be the group ring $\mathbb{Z}G$, with the usual augmentation $\mathbb{Z} \leftarrow \mathbb{Z}G$. Then construct F_i and a chain map $F_i \to A_i(i)$ by applying the previous lemma inductively. □

Corollary 5.10. *Suppose that $G \curvearrowright X$ is a free simplicial action of a group G on a metric simplicial complex X. Suppose that we have a system of (nonempty) G-invariant simplicial subcomplexes $X(0) \subset X(1) \subset \ldots \subset X(N)$ so that:*

(a) $X(i)/G$ is compact for each i,

(b) The induced mappings $\tilde{H}_i(X(k)) \to \tilde{H}_i(X(k+1))$ are zero for each $i \leq n \leq N$ and $0 \leq k < N$.

Then the group G is of type FP_n.

Proof. Apply Theorem 5.9 to $A_*(i) := C_*(X(i))$. □

Note that the above corollary is the converse to Corollary 5.2. Thus

Corollary 5.11. *Suppose that $G \curvearrowright X$ is a group action on a uniformly acyclic bounded geometry metric simplicial complex, $K := G(\star)$, where $\star \in X$. Then G is of type FP if and only if the the direct system of reduced homology groups $\{\tilde{H}_*(N_R(K))\}$ is pro-zero.*

Combining Theorem 5.9 and Lemma 5.1 we get:

Corollary 5.12. *Suppose for $i = 0, \ldots, 2n+1$ we have an augmented chain complex $A_*(i)$ of finitely generated projective $\mathbb{Z}G$-modules, and for $i = 1, \ldots, 2n+1$ we have augmentation preserving G-equivariant chain maps $a_i : A_*(i-1) \to A_*(i)$ which induce zero on reduced homology in dimensions $\leq n$. Then:*

1. There is a partial resolution F_*:

$$0 \leftarrow \mathbb{Z} \leftarrow F_0 \leftarrow \ldots \leftarrow F_n$$

by finitely generated free $\mathbb{Z}G$-modules and a G-equivariant chain mapping $f_* : F_* \to A_*(n)$. In particular G is of type FP_n.

2. For any $\mathbb{Z}G$-module M, the map $H^i(f) : H^i(A_*(n); M) \to H^i(F_*; M)$ carries the image $Im(H^i(A(2n); M) \to H^i(A(n); M))$ isomorphically onto $H^i(F_*; M)$ for $i = 0, \dots n-1$.

3. The map $H_i(f) : H_i(P_*; M) \to H_i(A_*(2n); M)$ is an isomorphism onto the image of $H_i(A_*(n); M) \to H_i(A_*(2n); M)$.

We now discuss a relative version of Corollaries 5.4 and 5.12. Let X be a uniformly acyclic bounded geometry metric simplicial complex, and G be a group acting freely simplicially on X; thus G has finite cohomological dimension since X is acyclic and $dim(X) < \infty$. Let $K \subset X$ be a G-invariant subcomplex so that K/G is compact; and let $\{C_\alpha\}_{\alpha \in I}$ be the deep components of $X - K$. Define $Y_R := \overline{X - N_R(K)}$, $Y_{\alpha,R} := C_\alpha \cap Y_R$. We will assume that the system

$$\{\tilde{H}_j(Y_{\alpha,R})\}_{R \geq 0}$$

is approximately zero for each j, α. In particular, $\{\tilde{H}_0(Y_{\alpha,R})\}_{R\geq 0}$ is approximately zero, which implies that each C_α is stable. Let H_α denote the stabilizer of C_α in G. Choose a set of representatives $C_{\alpha_1}, \dots, C_{\alpha_k}$ from the G-orbits in the collection $\{C_\alpha\}$. For notational simplicity we relabel $\alpha_1, \dots, \alpha_k$ as $1, \dots, k$. Let $H_i = H_{\alpha_i}$ be the stabilizer of $C_i = C_{\alpha_i}$. This defines a group pair $(G; H_1, ..., H_k)$. Let P_* be a finite length projective resolution of $\mathbb{Z}$ by $\mathbb{Z}G$-modules, and for each $i = 1, \dots, k$, we choose a finite length projective resolution of $\mathbb{Z}$ by $\mathbb{Z}H_i$-modules $Q_*(i)$. Using the construction described in section 3 (see the discussion of the group pairs) we convert this data to a pair (C_*, D_*) of finite length projective resolutions (consisting of $\mathbb{Z}G$-modules). We recall that D_* decomposes in a natural way as a direct sum $\oplus_\alpha D_*(\alpha)$ where each $D(\alpha)$ is a resolution of $\mathbb{Z}$ by projective $\mathbb{Z}H_\alpha$-modules. Now construct a $\mathbb{Z}H_i$-chain mapping $C_*(Y_{\alpha_i,0}) \to D_*(\alpha_i)$ using the acyclicity of $D_*(\alpha_i)$. We then extend this G-equivariantly to a mapping $C_*(Y_0) \to D_*$, and then to a $\mathbb{Z}G$-chain mapping $\rho_0 : (C_*(X), C_*(Y_0)) \to (C_*, D_*)$. By restriction, this defines a morphism of inverse systems $\rho_R : (C_*(X), C_*(Y_R)) \to (C_*, D_*)$.

Lemma 5.13. *The mapping $\rho_\bullet$ induces approximate isomorphisms between relative (co)homology with local coefficients:*

$$H^*(G, \{H_i\}; M) \to H^*(C_*(X), C_*(Y_R); M) \simeq H^*(X/G, Y_R/G; M)$$

$$H_*(X/G, Y_R/G; M) \simeq H_*(C_*(X), C_*(Y_R); M) \to H_*(G, \{H_i\}; M)$$

for any $\mathbb{Z}G$-module M.

Proof. We will prove the lemma by showing that the maps ρ_R form an "approximate chain homotopy equivalence" in an appropriate sense.

For each i we construct a $\mathbb{Z}H_i$-chain mapping $D_*(i) \to C_*(Y_{i,R})$ using part 1 of Lemma 5.1 and the fact that

$$\{\tilde{H}_j(Y_{\alpha,R})\}_{R \geq 0}$$

is an approximately zero system. We then extend these to $\mathbb{Z}G$-chain mappings

$$f_R : (C_*, D_*) \to (C_*(X), C_*(Y_R)).$$

Using part 2 of Lemma 5.1, we can actually choose the mappings f_R so that they form a compatible system chain mappings up to chain-homotopy. The composition

$$\rho_R \circ f_R : (C_*, D_*) \to (C_*, D_*)$$

is $\mathbb{Z}G$-chain mapping, hence it is chain-homotopic to the identity. The composition

$$f_R \circ \rho_R : C_*(X, Y_R) \to C_*(X, Y_R)$$

need not be chain homotopic to the identity, but it becomes chain homotopic to the projection map when precomposed with the restriction $C_*(X, Y_{R'}) \to C_*(X, Y_R)$ where $R' \geq R$ is suitably chosen (by again using part 2 of Lemma 5.1 and the fact that

$$\{\tilde{H}_j(Y_{\alpha,R})\}_{R\geq 0}$$

is an approximately zero system). This clearly implies the induced homorphisms on (co)homology are approximate isomorphisms. □

6. Coarse Poincare duality

We now introduce a class of metric simplicial complexes which satisfy coarse versions of Poincare and Alexander duality, see Theorems 6.7, 7.5, 7.7.

From now on we will adopt the convention of extending each (co)chain complex indexed by the nonnegative integers to a complex indexed by the integers by setting the remaining groups equal to zero. So for each (co)chain complex $\{C_i, i \geq 0\}$ we get the (co)homology groups $H_i(C_*), H^i(C_*)$ defined for $i < 0$.

Definition 6.1 (Coarse Poincaré duality spaces). A *coarse Poincaré duality space of formal dimension* n is a bounded geometry metric simplicial complex X so that $C_*(X)$ is uniformly acyclic, and there is a constant D_0 and chain mappings

$$C_*(X) \xrightarrow{\bar{P}} C_c^{n-*}(X) \xrightarrow{P} C_*(X)$$

so that

1. P and $\bar{P}$ have displacement $\leq D_0$ (see section 2 for the definition of displacement).

2. $\bar{P} \circ P$ and $P \circ \bar{P}$ are chain homotopic to the identity by D_0-Lipschitz[8] chain homotopies $\Phi : C_*(X) \to C_{*+1}(X)$, $\bar{\Phi} : C_c^*(X) \to C_c^{*-1}(X)$.

We will often refer to coarse Poincare duality spaces of formal dimension n as *coarse* $PD(n)$ *spaces.* Throughout the paper we will reserve the letter D_0 for the constant which appears in the definition of a coarse $PD(n)$ space; we let $D := D_0 + 1$.

Note that for each coarse $PD(n)$ space X we have

$$H_c^*(X) \simeq H_{n-*}(X) \simeq H_{n-*}(\mathbb{R}^n) \simeq H_c^*(\mathbb{R}^n).$$

We will not need the bounded geometry and uniform acyclicity conditions until Theorem 7.7. Later in the paper we will consider simplicial actions on coarse $PD(n)$ spaces, and we will assume implicitly that the actions commute with the operators $\bar{P}$ and P, up to chain homotopy with uniformly bounded Lipschitz constants.

The next lemma gives important examples of coarse $PD(n)$ spaces:

[8] See section 2.

Lemma 6.2. *The following are coarse $PD(n)$ spaces:*

1. An acyclic metric simplicial complex X which admits a free, simplicial, cocompact action by a $PD(n)$ group.

2. An n-dimensional, bounded geometry metric simplicial complex X, with an augmentation $\alpha : C_c^n(X) \to \mathbb{Z}$ for the compactly supported simplicial cochain complex, so that $(C_c^*(X), \alpha)$ is uniformly acyclic (see section 2 for definitions).

3. A uniformly acyclic, bounded geometry metric simplicial complex X which is a topological n-manifold.

Proof of 1. Let $0 \leftarrow \mathbb{Z} \leftarrow P_0 \leftarrow \ldots \leftarrow P_n \leftarrow 0$ be a resolution of $\mathbb{Z}$ by finitely generated projective $\mathbb{Z}G$-modules. X is acyclic, so we have $\mathbb{Z}G$-chain homotopy equivalences $P_* \overset{\alpha}{\simeq} C_*(X)$ and $Hom(P_*, \mathbb{Z}G) \simeq C_c^*(X)$ where α is augmentation preserving. Hence to construct the two chain equivalences needed in Definition 6.1, it suffices to construct a $\mathbb{Z}G$-chain homotopy equivalence $p : P_* \to Hom(P_{n-*}, \mathbb{Z}G)$ of $\mathbb{Z}G$-modules (since the operators are G-equivariant conditions 1 and 2 of Definition 6.1 will be satisfied automatically). For this, see [12, p. 221].

Proof of 2. We construct a chain mapping $P : C_*(X) \to C_c^{n-*}(X)$ as follows. We first map each vertex v of X to an n-cocycle $\beta \in C_c^n(X, \overline{X - B(v, R_0)})$ which maps to 1 under the augmentation α, (such a β exists by the uniform acyclicity of $(C_c^*(X), \alpha)$), and extend this to a homomorphism $C_0(X) \to C_c^n(X)$. By the uniform acyclicity of $(C_c^*(X), \alpha)$ we can extend this to a chain mapping P. By similar reasoning we obtain a chain homotopy inverse $\bar{P}$, and construct chain homotopies $\bar{P} \circ P \sim id$ and $P \circ \bar{P} \sim id$.

Proof of 3. X is acyclic, and therefore orientable. An orientation of X determines an augmentation $\alpha : C_c^n(X) \to \mathbb{Z}$. The uniform acyclicity of X together with ordinary Poincare duality implies that $(C_c^*(X), \alpha)$ is uniformly acyclic. So 3 follows from 2.

We remark that if $G \curvearrowright X$ is a free simplicial action then these constructions can be made G-invariant. $\square$

When $K \subset X$ is a (nonempty) subcomplex we will consider the direct system of tubular neighborhoods $\{N_R(K)\}_{R \geq 0}$ of K and the inverse system of the closures of their complements

$$\{Y_R := \overline{X - N_R(K)}\}_{R \geq 0}.$$

We get four inverse and four direct systems of (co)homology groups:

$$\{H_c^k(N_R(K))\},\ \{H_j(X, Y_R)\},\ \{H_c^k(X, N_R(K))\},\ \{H_j(Y_R)\}$$

$$\{H_c^k(Y_R)\},\ \{H_j(X, N_R(K))\},\ \{H_c^k(X, Y_R)\},\ \{H_j(N_R(K))\}$$

with the usual restriction and projection homomorphisms. Note that by excision, we have isomorphisms

$$H_j(X, Y_R) \simeq H_j(N_R(K), \partial N_R(K)), \quad \text{etc.}$$

Extension by zero defines a group homomorphism $C_c^k(N_{R+D}(K)) \overset{ext}{\subset} C_c^k(X)$. When we compose this with

$$C_c^k(X) \overset{P}{\to} C_{n-k}(X) \overset{proj}{\to} C_{n-k}(X, Y_R)$$

we get a well-defined induced homomorphism

$$P_{R+D} : H^k_c(N_{R+D}(K)) \to H_{n-k}(X, Y_R)$$

where D is as in Definition 6.1. We get, in a similar fashion, homomorphisms

$$H^k_c(N_{R+D}(K)) \xrightarrow{P_{R+D}} H_{n-k}(X, Y_R) \xrightarrow{\bar{P}_R} H^k_c(N_{R-D}(K)) \tag{6.3}$$

$$H^k_c(Y_R) \xrightarrow{P_R} H_{n-k}(X, N_{R+D}(K)) \xrightarrow{\bar{P}_{R+D}} H^k_c(Y_{R+2D}) \tag{6.4}$$

$$H^k_c(X, N_{R+D}(K)) \xrightarrow{P_{R+D}} H_{n-k}(Y_R) \xrightarrow{\bar{P}_R} H^k_c(X, N_{R-D}(K)) \tag{6.5}$$

$$H^k_c(X, Y_R) \xrightarrow{P_R} H_{n-k}(N_{R+D}(K)) \xrightarrow{\bar{P}_{R+D}} H^k_c(X, Y_{R+2D}) \tag{6.6}$$

Note that the homomorphisms in (6.3), (6.5) determine α-morphisms between inverse systems and the homomorphisms in (6.4), (6.6) determine β-morphisms between direct systems, where $\alpha(R) = R - D$, $\beta(R) = R + D$ (see section 4 for definitions). These operators inherit the bounded displacement property of P and $\bar{P}$, see condition 1 of Definition 6.1. We let $\omega(R) := R + 2D$, where D is the constant from Definition 6.1.

Theorem 6.7 (Coarse Poincare duality). *Let X be a coarse $PD(n)$ space, $K \subset X$ be a subcomplex as above. Then the morphisms $P_\bullet, \bar{P}_\bullet$ in (6.3), (6.5) are (ω, ω)-approximate isomorphisms of inverse systems and the morphisms $P_\bullet, \bar{P}_\bullet$ in (6.4), (6.6) are (ω, ω)-approximate isomorphisms of direct systems (see section 4). In particular, if $X \neq N_{R_0}(K)$ for any R_0 then the inverse systems $\{H^n_c(N_R(K))\}_{R\geq 0}$ and $\{H_n(Y_R)\}_{R\geq 0}$ are approximately zero.*

Proof. We will verify the assertion for the homomorphism $P_\bullet$ in (6.3) and leave the rest to the reader. We first check that $P_\bullet$ is an ω-approximate monomorphism. Let

$$\xi \in Z^*_c(N_{R+2D}(K))$$

be a cocycle representing an element $[\xi] \in Ker(P_{R+2D})$, and let $\xi_1 \in C^*_c(X)$ be the extension of ξ by zero. Then we have

$$P(\xi_1) = \partial \eta + \zeta$$

where $\eta \in C_{n-*}(X)$ and $\zeta \in C_{n-*}(\overline{X - N_{R+D}(K)})$. Applying $\bar{P}$ and the chain homotopy Φ, we get

$$\delta\Phi(\xi_1) + \Phi\delta(\xi_1) = \bar{P} \circ P(\xi_1) - \xi_1 = \bar{P}(\partial\eta + \zeta) - \xi_1$$

so

$$\xi_1 = \delta\bar{P}(\eta) + \bar{P}(\zeta) - \delta\Phi(\xi_1) - \Phi\delta(\xi_1).$$

The second and fourth terms on the right hand side vanish upon projection to $H^*_c(N_R(K))$, so $[\xi] \in Ker(H^*_c(N_{R+2D}(K)) \to H^*_c(N_R(K))$.

We now check that $P_\bullet$ is an ω-approximate epimorphism. Let

$$[\sigma] \in Im(H_{n-*}(X, \overline{X - N_{R+2D}(K)}) \to H_{n-*}(X, \overline{X - N_R(K)})),$$

then σ lifts to a chain $\tau \in C_{n-*}(X)$ so that $\partial\tau \in C_{n-*}(\overline{X - N_{R+2D}(K)})$. Let $[\tau] \in H_{n-*}(X, Y_{R+2D})$ be the corresponding relative homology class. Applying P and the chain homotopy $\bar{\Phi}$, we get

$$P(\bar{P}(\tau)) - \tau = \partial\bar{\Phi}(\tau) + \bar{\Phi}(\partial\tau).$$

Since $\bar{\Phi}(\partial\tau)$ vanishes in $C_{n-*}(X, \overline{X - N_R(K)})$, we get that

$$[\sigma] = P_{R+D}(\bar{P}_{R+2D}([\tau])).$$

The proof of the last assertion about $\{H_c^n(N_R(K))\}_{R\geq 0}$ and $\{H_n(Y_R)\}_{R\geq 0}$ follows since they are approximately isomorphic to zero systems $H_0(X, Y_R)$ and $H^0(X, N_R(K))$. $\square$

7. Coarse Alexander duality and coarse Jordan separation

In this section as in the previous one, we extend complexes indexed by the nonnegative integers to complexes indexed by $\mathbb{Z}$, by setting the remaining groups equal to zero.

Let X, K, D, Y_R, and ω be as in the preceding section. Composing the morphisms $P_\bullet$ and $\bar{P}_\bullet$ with the boundary operators for long exact sequences of pairs, we obtain the compositions A_{R+D}

$$H_c^*(N_{R+D}(K)) \xrightarrow{P_{R+D}} H_{n-*}(X, Y_R) \overset{\partial}{\simeq} \tilde{H}_{n-*-1}(Y_R) \tag{7.1}$$

and $\bar{A}_{R+D}$

$$\tilde{H}_{n-*-1}(Y_{R+D}) \overset{\partial^{-1}}{\simeq} H_{n-*}(X, Y_{R+D}) \xrightarrow{\bar{P}_{R+D}} H_c^*(N_R(K)). \tag{7.2}$$

Similarly, composing the maps from (6.3)-(6.4) with boundary operators and their inverses, we get:

$$H_c^*(Y_R) \xrightarrow{A_R} \tilde{H}_{n-*-1}(N_{R+D}(K)) \tag{7.3}$$

and

$$\tilde{H}_{n-*-1}(N_R(K)) \xrightarrow{\bar{A}_R} H_c^*(Y_{R+D}). \tag{7.4}$$

Theorem 7.5 (Coarse Alexander duality). *1. The morphisms $A_\bullet$ and $\bar{A}_\bullet$ in (7.1)-(7.4) are (ω, ω)-approximate isomorphisms.*

2. The maps $A_\bullet$ in (7.1) and (7.3) have displacement at most D. The map $\bar{A}_\bullet$ in (7.2) (respectively (7.4)) has displacement at most D in the sense that if $\sigma \in Z_{n-*-1}(Y_{R+D})$ ($\sigma \in Z_{n-*-1}(N_R(K))$), and $\sigma = \partial\tau$ for $\tau \in C_{n-*}(X)$, then the support of $\bar{A}_{R+D}([\sigma])$ (respectively $\bar{A}_R([\sigma])$) is contained in $N_D(Support(\tau))$.

Like ordinary Alexander duality, this theorem follows directly from Theorem 6.7, and the long exact sequence for pairs.

Combining Theorem 7.5 with Corollary 5.4 we obtain:

Theorem 7.6 (Coarse Alexander duality for FP_k groups). *Let X be a coarse $PD(n)$ space, and let G, P_*, $G \curvearrowright X$, f, and K be as in the statement of Corollary 5.4. Then*

1. The family of compositions

$$\tilde{H}_{n-i-1}(Y_{R+D}) \xrightarrow{\bar{A}} H^i_c(N_R(K)) \xrightarrow{f^i_R} H^i(P_*;\mathbb{Z}G)$$

defines an approximate isomorphism when $i < k$, and an approximate monomorphism when $i = k$. Recall that for $i < k$ we have a natural isomorphism $H^i(P_*,\mathbb{Z}G) \simeq H^i(G,\mathbb{Z}G)$.

2. The family of compositions

$$\tilde{H}_i(P_*;\mathbb{Z}G) \to \tilde{H}_i(N_R(K)) \xrightarrow{\bar{A}_R} H^{n-i-1}_c(Y_{R+D})$$

is an approximate isomorphism when $i < k$, and an approximate epimorphism when $i = k$. Recall that $\tilde{H}_i(P_*;\mathbb{Z}G) = \{0\}$ for $i < k$ since G is of type FP_k.

Theorem 7.7 (Coarse Alexander duality for maps). *Suppose X is a coarse $PD(n)$ space, X' is a bounded geometry uniformly $(k-1)$-acyclic metric simplicial complex, and $f : C_*(X') \to C_*(X)$ is a uniformly proper chain map. Let $K := Support(f(C_*(X'))$, $Y_R := \overline{X - N_R(K)}$. Then:*

1. The family of compositions

$$\tilde{H}_{n-i-1}(Y_{R+D}) \xrightarrow{\bar{A}} H^i_c(N_R(K)) \xrightarrow{H^i_c(f_R)} H^i_c(X')$$

defines an approximate isomorphism when $i < k$, and an approximate monomorphism when $i = k$.

2. The family of compositions

$$\tilde{H}_i(X') \to \tilde{H}_i(N_R(K)) \xrightarrow{\bar{A}_R} H^{n-i-1}_c(Y_{R+D})$$

is an approximate isomorphism when $i < k$, and an approximate epimorphism when $i = k$.[9]

3. Furthermore, these approximate isomorphisms approximately respect support in the following sense. There is a function $\zeta : \mathbb{N} \to \mathbb{N}$ so that if $i < k$, $S \subset X'$ is subcomplex, $T := Support(f_*(C_*(S))) \subset X$ is the corresponding subcomplex of X, and $\alpha \in Im(H^i_c(X', \overline{X' - S}) \to H^i_c(X'))$, then α belongs to the image of the composition

$$\tilde{H}_{n-i-1}(Y_R \cap N_{\zeta(R)}(T)) \to \tilde{H}_{n-i-1}(Y_R) \xrightarrow{H^i_c(f)\circ\bar{A}} H^i_c(X').$$

4. If $k = n+1$, then $H^n_c(X') = \{0\}$ unless $N_R(K) = X$ for some R.

Proof. Parts 1, 2 and 3 of Theorem follow from Lemma 5.5 and Theorem 7.5. Part 4 follows since for $i = n$, $\{\tilde{H}_{n-i-1}(Y_{R+D})\} = \{0\}$ is approximately isomorphic to the constant system $\{H^n_c(X')\}$. □

We now give a number of corollaries of coarse Alexander duality.

[9]The function ω for the above approximate isomorphisms depends only on the distortion of f, the acyclicity functions for X and X', and the bounds on the geometry of X and X'.

Corollary 7.8 (Coarse Jordan separation for maps). *Let X and X' be n-dimensional and $(n-1)$-dimensional coarse Poincaré duality spaces respectively, and let $g : X' \to X$ be a uniformly proper simplicial map. Then*

1. $g(X')$ coarsely separates X into (exactly) two components.

2. For every R, each point of $N_R(g(X'))$ lies within uniform distance from each of the deep components of $Y_R := \overline{X - N_R(g(X'))}$.

3. If $Z \subset X'$ and $X' \not\subset N_R(Z)$ for any R, then $g(Z)$ does not coarsely separate X.

Proof. We have the following diagram:

$$\begin{array}{ccc} \tilde{H}_0(Y_R) & \xrightarrow{H_c^{n-1}(g)\circ \bar{A}} & H_c^{n-1}(X') = \mathbb{Z} \\ \uparrow & & \\ \lim\limits_{\overleftarrow{R}} \tilde{H}_0^{Deep}(Y_R) & & \end{array}$$

where the family of morphisms $H_c^{n-1}(g)\circ \bar{A}$ gives rise to an approiximate isomorphism. Thus

$$\lim_{\overleftarrow{R}} \tilde{H}_0^{Deep}(Y_R) = \mathbb{Z}$$

which implies 1. Let $x \in N_R(K)$. Then there exists a representative α of a generator of $H_c^{n-1}(X')$ such that $H_c^{n-1}(g)(\alpha) \in C_c^{n-1}(X)$ is supported uniformly close to x. We apply Part 3 of Theorem 7.7 to the class $[H_c^{n-1}(g)(\alpha)]$ to prove 2. □

As a special case of the above corollary we have:

Corollary 7.9 (Coarse Jordan separation for submanifolds). *Let X and X' be n-dimensional and $(n-1)$-dimensional uniformly acyclic PL-manifolds respectively, and let $g : X' \to X$ be a uniformly proper simplicial map. Then the assertions 1, 2 and 3 from the preceeding theorem hold.*

Similarly to the Corollary 7.8 we get:

Corollary 7.10 (Coarse Jordan separation for groups). *Let X be a coarse $PD(n)$-space and G be a $PD(n-1)$-group acting freely simplicially on X. Let $K \subset X$ be a G-invariant subcomplex with K/G compact. Then:*

1. G coarsely separates X into (exactly) two components.

2. For every R, each point of $N_R(K)$ lies within uniform distance from each of the deep components of $\overline{X - N_R(K)}$.

Lemma 7.11. *Let W be a bounded geometry metric simplicial complex which is homeomorphic to a union of $W = \cup_{i\in I} W_i$ of k half-spaces $W_i \simeq \mathbb{R}_+^{n-1}$ along their boundaries. Assume that for $i \neq j$, the union $W_i \cup W_j$ is uniformly acyclic and is uniformly properly embedded in W. Let $g : W \to X$ be a uniformly proper map of W into a coarse $PD(n)$ space X. Then $g(W)$ coarsely separates X into k components. Moreover, there is a unique cyclic ordering on the index set I so that for R sufficiently large, the frontier of each deep component C of $X - N_R(g(W))$ is at finite Hausdorff distance from $g(W_i) \cup g(W_j)$ where i and j are adjacent with respect to the cyclic ordering.*

Proof. We have $H_c^{n-1}(W) \simeq \mathbb{Z}^{k-1}$, so, arguing analogously to Corollary 7.8, we see that $g(W)$ coarsely separates X into k components. Applying coarse Jordan separation and the fact that no W_i coarsely separates W_j in W, we can define the desired cyclic ordering by declaring that i and j are consecutive iff $g(W_i) \cup g(W_j)$ coarsely separates X into two deep components (Corollary 7.8), one of which is a deep component of $X - g(W)$. We leave the details to the reader. □

Lemma 7.12. *Suppose G is a group of type FP_{n-1} of cohomological dimension $\leq n-1$, and let P_*, f, $G \curvearrowright X$, $K \subset X$ and Y_R be as in Theorem 7.6. Then every deep component of Y_R is stable for $R \geq D$; in particular, there are only finitely many deep components of Y_R modulo G. If $dim(G) < n-1$ then there is only one deep component.*

Proof. The composition

$$\lim_{\overleftarrow{R}} \tilde{H}_0^{Deep}(Y_R) \to \tilde{H}_0^{Deep}(Y_D) \xrightarrow{f_D^i \circ \bar{A}_D} H^{n-1}(P_*; \mathbb{Z}G) \tag{7.13}$$

is an isomorphism by Theorem 7.6. Therefore

$$\tilde{H}_0^{Deep}(Y_R) \to \tilde{H}_0^{Deep}(Y_D)$$

is a monomorphism for any $R \geq D$, and hence every deep component of Y_D is stable. If $dim(G) < n-1$ then $H^{n-1}(P_*, \mathbb{Z}G) = \{0\}$, and by (7.13) we conclude that Y_D contains only one deep component. □

Another consequence of coarse Jordan separation is:

Corollary 7.14. *Let $G \curvearrowright X$ be a free simplicial action of a group G of type FP on a coarse $PD(n)$ space X, and let $K \subset X$ be a G-invariant subcomplex on which G acts cocompactly. By Lemma 7.12 there is an R_0 so that all deep components of $X - N_{R_0}(K)$ are stable; hence we have a well-defined collection of deep complementary components $\{C_\alpha\}$ and their stabilizers $\{H_\alpha\}$. If $H \subset G$ is a $PD(n-1)$ subgroup, then one of the following holds:*

1. *H coarsely separates G.*
2. *H has finite index in G, and so G is a $PD(n-1)$ group.*
3. *H has finite index in H_α for some α.*

In particular, G contains only finitely many conjugacy classes of maximal, coarsely nonseparating $PD(n-1)$ subgroups.

Proof. We assume that H does not coarsely separate G. Pick a basepoint $\star \in K$, and let $W := H(\star)$ be the H-orbit of $\star$. Then by Corollary 7.10 there is an R_1 so that $X - N_{R_1}(W)$ has two deep components C_+, C_- and both are stable. Since H does not coarsely separate G, we may assume that $K \subset N_{R_2}(C_-)$ for some R_2. Therefore C_+ has finite Hausdorff distance from some deep component C_α of $X - N_{R_0}(K)$, and clearly the Hausdorff distance between the frontiers ∂C_+ and ∂C_α is finite. Either H preserves C_+ and C_-, or it contains an element h which exchanges the two. In the latter case, $h(C_\alpha)$ is within finite Hausdorff distance from C_-; so in this case K is contained in $N_r(W)$ for some r, and this implies 2. When H preserves C_+ then we have $H \subset H_\alpha$, and since H acts cocompactly on ∂C_+, it also acts cocompactly on ∂C_α and hence $[H_\alpha : H] < \infty$. □

8. The proofs of Theorems 1.1 and 1.3

Sketch of the proof of Theorem 1.1. Consider an action $G \curvearrowright X$ as in the statement of Theorem 1.1. Let $K \subset X$ be a G-invariant subcomplex with K/G compact. By Lemma 7.12 the deep components of $X - N_R(K)$ stabilize at some R_0, and hence we have a collection of deep components C_α and their stabilizers H_α. Naively one might hope that for some $R \geq R_0$, the tubular neighborhood $N_R(K)$ is acyclic, and the frontier of $N_R(K)$ breaks up into connected components which are in one-to-one correspondence with the C_α's, each of which is acyclic and has the same compactly supported cohomology as $\mathbb{R}^{n-1}$. Of course, this is too much to hope for, but there is a coarse analog which does hold. To explain this we first note that the systems $\tilde{H}_*(N_R(K))$ and $H_c^*(N_R(K))$ are approximately zero and approximately constant respectively by Corollary 5.4. Applying coarse Alexander duality, we find that the systems $H_c^*(Y_R)$ and $\tilde{H}_*(Y_R)$ corresponding to the complements $Y_R := \overline{X - N_R(K)}$ are approximately zero and approximately constant, respectively. Instead of looking at the frontiers of the neighborhoods $N_R(K)$, we look at metric annuli $A(r, R) := \overline{N_R(K) - N_r(K)}$ for $r \leq R$. One can try to compute the (co)homology of these annuli using a Mayer-Vietoris sequence for the covering $X = N_R(K) \cup Y_r$; however, the input to this calculation is only approximate, and the system of annuli does not form a direct or inverse system in any useful way. Nonetheless, there are finite direct systems of nested annuli of arbitrary depth for which one can understand the (co)homology, and this allows us[10] to apply results from section 5 to see that the H_α's are Poincare duality groups.

The proof of Theorem 1.1. We now assume that G is a group of type FP acting freely simplicially on a coarse $PD(n)$ space X. This implies that $dim(G) \leq n$, so by Lemma 3.2 there is a resolution $0 \to P_n \to \ldots \to P_0 \to \mathbb{Z} \to 0$ of $\mathbb{Z}$ by finitely generated projective $\mathbb{Z}G$-modules. We may construct G-equivariant (augmentation preserving) chain mappings $\rho : C_*(X) \to P_*$ and $f : P_* \to C_*(X)$ using the acyclicity of $C_*(X)$ and P_*; the composition $\rho \circ f : P_* \to P_*$ is $\mathbb{Z}G$-chain homotopic to the identity. If $L \subset X$ is a G-invariant subcomplex for which L/G is compact, then we get an induced homomorphism

$$H^*(G; \mathbb{Z}G) \xrightarrow{H^*(\rho)} H^*(X/G; \mathbb{Z}G) \to H^*(L/G; \mathbb{Z}G) \simeq H_c^*(L);$$

abusing notation we will denote this composition by $H^*(\rho)$.

Let $K \subset X$ be a connected, G-invariant subcomplex so that K/G is compact and the image of f is supported in K. For $R \geq 0$ set $Y_R := \overline{X - N_R(K)}$. Corollary 5.4 tells us that the families of maps

$$\{0\} \to \{\tilde{H}_*(P_*; \mathbb{Z}G)\} \to \{\tilde{H}_*(N_R(K))\} \tag{8.1}$$

$$H_c^*(f) : H_c^*(N_R(K)) \to H^*(G; \mathbb{Z}G) \simeq H^*(P; \mathbb{Z}G). \tag{8.2}$$

[10]There is an extra complication in calculating H_c^{n-1} for the annuli which we've omitting from this sketch.

define approximate isomorphisms. Applying Theorems 7.6 we get approximate isomorphisms

$$\{0\} \to H^k_c(Y_R) \text{ for all } k \tag{8.3}$$

and

$$\phi_{k,R} : \tilde{H}_k(Y_R) \to H^{n-k-1}(P_*;\mathbb{Z}G) \simeq H^{n-k-1}(G;\mathbb{Z}G) \text{ for all } k. \tag{8.4}$$

We denote $\phi_{*,D}$ by ϕ_*.

We now apply Lemma 7.12 to see that every deep component of $X - N_D(K)$ is stable. Let $\{C_\alpha\}$ denote the collection of deep components of $X - N_D(K)$, and set $Y_{R,\alpha} := Y_R \cap C_\alpha$ and $Z_{R,\alpha} := \overline{X - Y_{R,\alpha}}$. Note that for every α, and $D < r < R$ we have $Z_{R,\alpha} \cap Y_{r,\alpha} = \overline{N_R(K) - N_r(K)} \cap C_\alpha$.

Lemma 8.5. *1. There is an R_0 so that if $R \geq R_0$ then $Y_{R,\alpha} = \overline{X - Z_{R,\alpha}}$ and $Z_{R,\alpha} = N_{R-R_0}(Z_{R_0,\alpha})$.*

2. The systems $\{\tilde{H}_k(Y_{R,\alpha})\}$, $\{\tilde{H}_k(Z_{R,\alpha})\}$, $\{H^k_c(Y_{R,\alpha})\}$, $\{H^k_c(Z_{R,\alpha})\}$ are approximately zero for all k.

Proof. Pick R_0 large enough that all shallow components of $X - N_D(K)$ are contained in $N_{R_0-1}(K)$. Then for all $R \geq R_0$, $\partial C_\alpha \cap Y_R = \emptyset$ and hence $Y_{R,\alpha}$, like Y_R itself, is the closure of its interior; this implies that $Y_{R,\alpha} = \overline{X - \overline{X - Y_{R,\alpha}}} = \overline{X - Z_{R,\alpha}}$. We also have $Z_{R,\alpha} = N_R(K) \sqcup (\sqcup_{\beta\neq\alpha} C_\beta)$ for all $R \geq R_0$. Since $\sqcup_{\beta\neq\alpha} N_R(C_\beta) \subset N_{R_0+R}(K) \cup (\sqcup_{\beta\neq\alpha} C_\beta)$, we get

$$\begin{aligned} N_R(Z_{R_0,\alpha}) &= N_{R_0+R}(K) \cup (\sqcup_{\beta\neq\alpha} N_R(C_\beta)) \\ &= N_{R_0+R}(K) \cup (\sqcup_{\beta\neq\alpha} C_\beta) \\ &= Z_{R_0+R,\alpha}. \end{aligned}$$

Thus we have proven 1.

To prove 2, we first note that $\{\tilde{H}_0(Y_{R,\alpha})\}$ is approximately zero by the stability of the deep components C_α. When $R \geq R_0$ then $Z_{R,\alpha}$ is connected (since $N_R(K)$ and each C_β are connected), and this says that $\{\tilde{H}_0(Z_{R,\alpha})\}$ is approximately zero. When $R \geq R_0$ then Y_R is the disjoint union $\sqcup_\alpha Y_{R,\alpha}$, so we have direct sum decompositions $H_k(Y_R) = \oplus_\alpha H_k(Y_{R,\alpha})$ and $H^k_c(Y_R) = \oplus_\alpha H^k_c(Y_{R,\alpha})$ which are compatible projection homomorphisms. This together with (8.3) and (8.4) implies that $\{\tilde{H}_k(Y_{R,\alpha})\}$ and $\{H^k_c(Y_{R,\alpha})\}$ are approximately zero for all k. By part 1 and Theorem 7.5 we get that $\{H^k_c(Z_{R,\alpha})\}$ and $\{\tilde{H}_k(Z_{R,\alpha})\}$ are approximately zero for all k. ☐

Lemma 8.6. *There is an $R_{min} > D$ so that for any $R \geq R_{min}$ and any integer M, there is a sequence $R \leq R_1 \leq R_2 \leq ... \leq R_M$ with the following property. Let $A(i,j) := \overline{N_{R_j}(K) - N_{R_i}(K)} \subset Y_{R_i}$, and $A_\alpha(i,j) := A(i,j) \cap C_\alpha$. Then for each $1 < i < j < M$,*

1. The image of $\tilde{H}_k(A(i,j)) \to \tilde{H}_k(A(i-1,j+1))$ maps isomorphically onto $H^{n-k-1}(G;\mathbb{Z}G)$ under the composition $\tilde{H}_k(A(i-1,j+1)) \to \tilde{H}_k(Y_D) \xrightarrow{\phi_k} H^{n-k-1}(G;\mathbb{Z}G)$ for $0 \leq k \leq n-1$. The homomorphism $\tilde{H}_n(A(i,j)) \to \tilde{H}_n(A(i-1,j+1))$ is zero.

2. $H^k(\rho) : H^k(G;\mathbb{Z}G) \to H^k_c(A(i,j))$ *maps* $H^k(G;\mathbb{Z}G)$ *isomorphically onto the image of* $H^k_c(A(i-1,j+1)) \to H^k_c(A(i,j))$ *for* $0 \le k < n-1$.

3. There is a system of homomorphisms $H^{n-1}_c(A_\alpha(i,j)) \xrightarrow{\theta^\alpha_{i,j}} \mathbb{Z}$ *(compatible with the inclusions* $A_\alpha(i,j) \to A_\alpha(i-1,j+1)$*) so that the image of* $H^{n-1}_c(A_\alpha(i-1,j+1)) \to H^{n-1}_c(A_\alpha(i,j))$ *maps isomorphically to* $\mathbb{Z}$ *under* $\theta^\alpha_{i,j}$.

4. For each α, $\tilde H_0(A_\alpha(i,j)) \xrightarrow{0} \tilde H_0(A_\alpha(i-1,j+1))$.

Proof. We choose R_{min} large enough that for any $R \ge R_{min}$, the following inductive construction is valid. Let $R_1 := R$. Using the approximate isomorphisms (8.1), (8.2), (8.3), (8.4), and Lemma 8.5, we inductively choose R_{i+1} so that:

A. $\tilde H_k(N_{R_i}(K)) \xrightarrow{0} \tilde H_k(N_{R_{i+1}}(K))$ for $0 \le k \le n$.

B. $Im(\tilde H_k(Y_{R_{i+1}}) \to \tilde H_k(Y_{R_i}))$ maps isomorphically to $H^{n-k-1}(G;\mathbb{Z}G)$ under ϕ_{k,R_i} for $0 \le k < n$, and $Im(\tilde H_k(Y_{R_{i+1}}) \to \tilde H_k(Y_{R_i}))$ is zero when $k = n$.

C. $Im(H^*_c(N_{R_{i+1}}(K)) \to H^*_c(N_{R_i}(K)))$ maps isomorphically onto $H^*(G;\mathbb{Z}G)$ under $H^*_c(f)$.

D. $H^*_c(Y_{R_i}) \xrightarrow{0} H^*_c(Y_{R_{i+1}})$.

E. For each α, $H^{n-1}_c(Y_{R_i,\alpha}) \xrightarrow{0} H^{n-1}_c(Y_{R_{i+1},\alpha})$, and $H^{n-1}_c(Z_{R_{i+1},\alpha}) \xrightarrow{0} H^{n-1}_c(Z_{R_i,\alpha})$.

F. For each α, $\tilde H_0(Y_{R_{i+1},\alpha}) \xrightarrow{0} \tilde H_0(Y_{R_i,\alpha})$ and $\tilde H_0(Z_{R_i,\alpha}) \xrightarrow{0} \tilde H_0(Z_{R_{i+1},\alpha})$.

Now take $1 < i < j < M$, and consider the map of Mayer-Vietoris sequences for the decompositions $X = N_{R_j}(K) \cup Y_{R_i}$ and $X = N_{R_{j+1}}(K) \cup Y_{R_{i-1}}$:

$$\begin{array}{ccccccc}
\tilde H_{k+1}(X) \to & \tilde H_k(A(i,j)) \to & \tilde H_k(N_{R_j}(K)) & \oplus & \tilde H_k(Y_{R_i}) & \to & \tilde H_k(X) \\
\downarrow & \downarrow & 0\downarrow & & \downarrow & & \downarrow \\
\tilde H_{k+1}(X) \to & \tilde H_k(A(i-1,j+1)) \to & \tilde H_k(N_{R_{j+1}}(K)) & \oplus & \tilde H_k(Y_{R_{i-1}}) & \to & \tilde H_k(X) \\
& \downarrow \phi_k\big|_{A(i-1,j+1)} & & & \downarrow \phi_k & & \\
& H^{n-k-1}(G,\mathbb{Z}G) \to & & & H^{n-k-1}(G,\mathbb{Z}G) & &
\end{array}$$

Since $\tilde H_*(X) = \{0\}$, conditions A and B and the diagram imply the first part of assertion 1. The same Mayer-Vietoris diagram for $k = n$ implies the second part.

Let $0 \le k < n-1$. Consider the commutative diagram of Mayer-Vietoris sequences:

$$\begin{array}{ccccccc}
& H^k(G,\mathbb{Z}G) \to & & & H^k(G,\mathbb{Z}G) & & \\
& H^k(\rho)\downarrow & & & H^k(\rho)\downarrow & & \\
H^k_c(X) \to & H^k_c(N_{R_{j+1}}(K)) & \oplus & H^k_c(Y_{R_{i-1}}) \to & H^k_c(A(i-1,j+1)) & \to & H^{k+1}_c(X) \\
\downarrow & \downarrow & & 0\downarrow & \downarrow & & \downarrow \\
H^k_c(X) \to & H^k_c(N_{R_j}(K)) & \oplus & H^k_c(Y_{R_i}) \to & H^k_c(A(i,j)) & \to & H^{k+1}_c(X)
\end{array}$$

Assertion 2 now follows from the fact that $H^k_c(X) \cong H^{k+1}_c(X) = 0$, conditions C and D, and the diagram.

Assertion 3 follows from condition E, the fact that $H^n_c(X) \simeq \mathbb{Z}$, and the following commutative diagram of Mayer-Vietoris sequences ($\theta^\alpha_{i,j}$ is the coboundary operator in the sequence):

$$\begin{array}{ccccc}
H_c^{n-1}(Z_{R_{j+1},\alpha}) \oplus H_c^{n-1}(Y_{R_{i-1},\alpha}) \to & H_c^{n-1}(A_\alpha(i-1,j+1)) & \xrightarrow{\theta_{i-1,j+1}} & H_c^n(X) \to 0 \\
0\downarrow \qquad\qquad 0\downarrow & \downarrow & & \downarrow \\
H_c^{n-1}(Z_{R_j,\alpha}) \oplus H_c^{n-1}(Y_{R_i,\alpha}) \to & H_c^{n-1}(A_\alpha(i,j)) & \xrightarrow{\theta_{i,j}} & H_c^n(X) \to 0
\end{array}$$

Assertion 4 follows from condition F and the following commutative diagram:

$$\begin{array}{ccccccc}
\tilde H_1(X) \to & \tilde H_0(A_\alpha(i,j)) \to & \tilde H_0(Z_{R_j,\alpha}) \oplus \tilde H_0(Y_{R_i,\alpha}) & \to & \tilde H_0(X) \\
\downarrow & \downarrow & 0\downarrow \qquad 0\downarrow & & \downarrow \\
\tilde H_1(X) \to & \tilde H_0(A_\alpha(i-1,j+1)) \to & \tilde H_0(Z_{R_{j+1},\alpha}) \oplus \tilde H_0(Y_{R_{i-1},\alpha}) & \to & \tilde H_0(X)
\end{array}$$

□

Corollary 8.7. *If G is an $(n-1)$-dimensional duality group, then each deep component stabilizer is a $PD(n-1)$ group.*

Proof. Fix a deep component C_α of $X - N_D(K)$, and let H_α be its stabilizer in G. Let $R = D$, $M = 4k+2$, and apply the construction of Lemma 8.6 to get $D \le R_1 \le R_2 \le \ldots \le R_{4k+2}$ satisfying the conditions of Lemma 8.6.

Pick $1 < i < j < M$. The mappings $\tilde H_\ell(A(i,j)) \to \tilde H_\ell(A(i-1,j+1))$ are zero for each $\ell = 1, ..., n$ by part 1 of Lemma 8.6, since $H^k(G, \mathbb{Z}G) = 0$ for $k < n-1$. Because $A(p,q)$ is the disjoint union $\amalg_\alpha A_\alpha(p,q)$ for all $0 < p < q < M$, we actually have $\tilde H_\ell(A_\alpha(i,j)) \xrightarrow{0} \tilde H_\ell(A_\alpha(i-1,j+1))$ for $1 \le \ell \le n$. By part 4 of Lemma 8.6 the same assertion holds for $\ell = 0$. Applying Theorem 5.9 to the chain complexes $C_*(A_\alpha(i,j))$, we see that when $k > 2n+5$, H_α is a group of type $FP(n)$. Since $dim(H_\alpha) \le dim(G) = n-1$ it follows that H_α is of type FP (see section 3).

The mappings $H_c^\ell(A_\alpha(i-1,j+1)) \to H_c^\ell(A_\alpha(i,j))$ are zero for $0 \le \ell < n-1$ by part 2 of Lemma 8.6 and the fact that $A(p,q) = \amalg_\alpha A_\alpha(p,q)$. By parts 1 and 2 of Lemma 5.1, we have $H^k(H_\alpha, \mathbb{Z}H_\alpha) = \{0\}$ for $0 \le k < n-1$, and $H^{n-1}(H_\alpha, \mathbb{Z}H_\alpha) \simeq \mathbb{Z}$ by part 3 of Lemma 8.6. Hence H_α is a $PD(n-1)$ group. □

Remark. For the remainder of the proof, we really only need to know that each deep component stabilizer is of type FP.

Proof of Theorem 1.1 concluded. Let $C_1, \ldots, C_k$ be a set of representatives for the G-orbits of deep components of $X - N_R(K)$, and let $H_1, \ldots, H_k \subset G$ denote their stabilizers. Since G and each H_i is of type FP, the group pair $(G, \{H_i\})$ has finite type (section 3). By Lemma 5.13, we have

$$H^*(G, \{H_i\}; \mathbb{Z}G) \simeq \lim_{\overrightarrow{R}} H_c^*(X, Y_R),$$

while $\lim_R H_c^*(X, Y_R) \simeq \lim_R H_{n-*}(N_R(K))$ by Coarse Poincare duality, and

$$\lim_{\overrightarrow{R}} H_*(N_R(K)) \simeq H_*(X) \simeq H_*(pt)$$

since homology commutes with direct limits. Therefore the group pair $(G, \{H_i\})$ satisfies one of the criteria for $PD(n)$ pairs (see section 3), and we have proven Theorem 1.1. □

We record a variant of Theorem 1.1 which describes the geometry of the action $G \curvearrowright X$ more explicitly:

Theorem 8.8. *Let $G \curvearrowright X$ be as in Theorem 1.1, and let $K \subset X$ be a G-invariant subcomplex with K/G compact. Then there are R_0, R_1, R_2 so that*

1. The deep components $\{C_\alpha\}_{\alpha \in I}$ of $X - N_{R_0}(K)$ are all stable, there are only finitely many of them modulo G, and their stabilizers $\{H_\alpha\}_{\alpha\in I}$ are $PD(n-1)$ groups.

2. For all $\alpha \in I$, the frontier ∂C_α is connected, and $N_{R_1}(\partial C_\alpha)$ has precisely two deep complementary components, E_α and F_α, where E_α has Hausdorff distance at most R_2 from C_α. Unless G is a $PD(n-1)$ group, the distance function $d(\partial C_\alpha, \cdot)$ is unbounded on $K \cap F_\alpha$.

3. The Hausdorff distance between $X - \amalg_\alpha E_\alpha$ and K is at most R_2.

Proof. This is clear from the discussion above. ☐

We remark that there are $\alpha_1 \neq \alpha_2 \in I$ so that the Hausdorff distance

$$d_H(\partial C_{\alpha_1}, \partial C_{\alpha_2}) < \infty$$

iff G is a $PD(n-1)$ group.

Lemma 8.9. *Let $G \curvearrowright X$ be as Theorem 1.1, and let K, C_α, H_α, C_i, H_i be as in the conclusion of the proof of Theorem 1.1. If X is simply connected and the groups H_i admit finite $K(H_i, 1)$'s, then G admits a finite $K(G, 1)$. There exists a contractible coarse $PD(n)$ space X' on which G acts freely and simplicially with the following properties:*

1. There is a G-equivariant proper homotopy equivalence $\phi : X \to X'$ which is a homeomorphism away from a finite tubular neighborhood of K.

2. There is a contractible subcomplex $K' \subset X'$ on which G acts cocompactly. All components of $X' - K'$ are deep and stable.

3. The mapping ϕ induces a bijection between the deep components C_α and components of $X' - K'$.

Proof. For each $1 \leq i \leq k$, let W_i be the universal cover of a finite Eilenberg-MacLane space for H_i, and specify an H_i-equivariant map $\psi_i : \partial C_i \to W_i$, where ∂C_i is the frontier of C_i. We can G-equivariantly identify the disjoint union $\amalg_{\alpha\in G(i)} \partial C_\alpha$ with the twisted product $G \times_{H_i} \partial C_i$, and obtain an induced G-equivariant mapping

$$\Psi : \cup_\alpha \partial C_\alpha = \cup_i (\amalg_{\alpha\in G(i)} \partial C_\alpha) \to \amalg_i (G \times_{H_i} W_i).$$

Let $K^+ := \overline{X - \amalg_\alpha C_\alpha}$. We now cut X open along the disjoint union $\partial C := \amalg_\alpha \partial C_\alpha$ to obtain a new complex

$$\ddot{X} := K^+ \amalg (\amalg_\alpha C_\alpha)$$

which contains two copies $\partial_+ C \subset K^+$ and $\partial_- C \subset \amalg_\alpha C_\alpha$ of ∂C. We let $\Psi_\pm$ be the corresponding copies of the mapping Ψ. Now define K' as the union (along $\partial_+ C$) of K^+ and the mapping cylinder of Ψ_+ and define Y' as the union (along $\partial_- C$) of $\amalg_\alpha C_\alpha$ and the mapping cylinder of Ψ_-. Finally obtain X' gluing K' and Y' along the copies of $W := \amalg_i (G \times_{H_i} W_i)$. The group G still acts on X' freely and simplicially and clearly K'/G is compact. By applying Van-Kampen's theorem and Mayer-Vietoris sequences, it follows that X' and K' are uniformly contractible. Assertion 1 is clear from the construction of X'. The remaining assertions follow easily from the first. $\square$

Proof of Theorem 1.3. By the main theorem deep components stabilizers H_i are $PD(2)$-groups and hence are surface groups by [17, 16]. Theorem 1.3 now follows by applying Lemma 8.9 (where the complexes W_i in the proof are homeomorphic to $\mathbb{R}^2$). $\square$

In Proposition 8.11 we generalize the uniqueness theorem of the peripheral structure from 3-dimensional manifolds to $PD(n)$ pairs.

Theorem 8.10. *(Jaco and Shalen [25], Johannson [26].) Let M be a compact connected acylindrical 3-manifold with aspherical incompressible boundary components $S_1, \ldots, S_m$. Let N be a compact 3-manifold homotopy-equivalent to M, with incompressible boundary components $Q_1, \ldots, Q_n$, and $\varphi : \pi_1(M) \to \pi_1(N)$ be an isomorphism. Then φ preserves the peripheral structures of $\pi_1(M)$ and $\pi_1(N)$ in the following sense. There is a bijection β between the set of boundary components of M and the set of boundary components on N so that after relabelling via β we have:*

$\varphi(\pi_1(S_i))$ is conjugate to $\pi_1(Q_i))$ in $\pi_1(N)$.

Proposition 8.11. *Let $(G, \{H_i\}_{i\in I})$ be a $PD(n)$ pair, where G is not a $PD(n-1)$ group, and H_i does not coarsely separate G for any i. Now let $G \curvearrowright X$ be a free simplicial action on a coarse $PD(n)$ space, and let $(G, \{L_j\}_{j\in J})$ be the group pair obtained by applying Theorem 1.1 to this action. Then there is a bijection $\beta : I \to J$ so that H_i is conjugate to $L_{\beta(i)}$ for all $i \in I$.*

Proof. Under the assumptions above, each H_i and L_j is a maximal $PD(n-1)$ subgroup (see Lemma 3.3). By Corollary 7.14, each H_i is conjugate to some L_j, and by Lemma 3.3 this defines an injection $\beta : I \to J$. Consider the double $\hat{G}$ of G over the L_j's. Then the double of G over the H_i's sits in $\hat{G}$, and the index will be infinite unless β is a bijection. $\square$

We now establish a relation between the acylindricity assumption in Theorem 8.10 and coarse nonseparation assumption in Proposition 8.11. We first note that if M is a compact 3-manifold with incompressible aspherical boundary components $S_1, \ldots, S_m$, then M is acylindrical iff $\pi_1(S_i) \cap g(\pi_1(S_j))g^{-1} = \{e\}$ whenever $i \neq j$ or $i = j$ but $g \notin \pi_1(S_i)$.

Lemma 8.12. *Suppose G is a duality group and $G \curvearrowright X$ is a free simplicial action on a coarse $PD(n)$ space, and let $(G, \{H_j\}_{j\in J})$ be the group pair obtained by applying Theorem 1.1 to this action. Assume that $H_i \cap (gH_jg^{-1}) = \{e\}$ whenever $i \neq j$ or $i = j$ but $g \notin H_i$. Then no H_i coarsely separates G.*

Proof. Let $K_0 \subset X$ be a connected G-invariant subcomplex so that K_0/G is compact and all deep components of $X - K_0$ are stable. Now enlarge K_0 to a subcomplex $K \subset X$ by throwing in the shallow (i.e. non-deep) components of $X - K_0$; then K is connected, G-invariant, K/G is compact, and all components of $X - K$ are deep and stable. Let $\{C_\alpha\}$ denote the components of $X - K$, and let C_i be a component stabilized by C_i. We will show that ∂C_i does not coarsely separate K in X. Since $K \hookrightarrow X$ is a uniformly proper embedding, $G \curvearrowright K$ is cocompact, and $H_i \curvearrowright \partial C_i$ is cocompact, this will imply the lemma.

For all components C_α and all R, the intersection $H_i \cap H_\alpha$ acts cocompactly on $N_R(\partial C_i) \cap \bar{C}_\alpha$, where H_α is the stabilizer of C_α; when $\alpha \neq i$ the group $H_i \cap H_\alpha$ is trivial, so in this case $Diam(N_R(\partial C_i) \cap \bar{C}_\alpha) < \infty$. For each R there are only finitely many α – modulo H_i – for which $N_R(\partial C_i) \cap C_\alpha \neq \emptyset$, so there is a constant $D_1 = D_1(R)$ so that if $\alpha \neq i$ then $Diam(N_R(\partial C_i) \cap C_\alpha) < D_1$. Each ∂C_α is connected and 1-ended, so we have an $R_1 = R_1(R)$ so that if $\alpha \neq i$, and $x, y \in \partial C_\alpha - N_{R_1}(\partial C_i)$, then x may be joined to y by a path in $\partial C_\alpha - N_R(\partial C_i)$.

By Corollary 7.10, there is a function $R_2 = R_2(R)$ so that if $x, y \in K - N_{R_2}(\partial C_i)$ then x may be joined to y by a path in $X - N_R(\partial C_i)$.

Pick R, and let $R' = R_2(R_1(R))$. If $x, y \in K - N_{R'}(\partial C_i)$ then they are joined by a path α_{xy} in $X - N_{R_1(R)}(\partial C_i)$. For each $\alpha \neq i$, the portion of α_{xy} which enters C_α may be replaced by a path in $\partial C_\alpha - N_R(\partial C_i)$. So x may be joined to y in $K - N_R(\partial C_i)$. Thus ∂C_i does not coarsely separate K in X. $\square$

Lemma 8.13. *Let M be a compact 3-manifold with $\partial M \neq \emptyset$, with aspherical incompressible nonempty boundary components $S_1, \ldots, S_m$. Then M is acylindrical if and only if $\pi_1(M)$ is not a surface group and no $H_i = \pi_1(S_i) \subset \pi_1(M) = G$ coarsely separates G.*

Proof. The implication $\Rightarrow$ follows from Lemma 8.12. To establish $\Leftarrow$ assume that M is not acylindrical. This implies that there exists a nontrivial decomposition of $\pi_1(M)$ as a graph of groups with a single edge group C which is a cyclic subgroup of some H_i. Thus C coarsely separates G. Since $[G : H_i] = \infty$ it follows that H_i coarsely separates G as well. $\square$

Corollary 8.14. *Suppose G is not a $PD(n-1)$ group, both $(G, \{H_i\}_{i\in I})$ and $(G, \{L_j\}_{j\in J})$ are $PD(n)$ pairs, no H_i coarsely separates G, and each L_j admits a finite Eilenberg-MacLane space. Then there is a bijection $\beta : I \to J$ so that H_i is conjugate to $L_{\beta(i)}$ for all $i \in I$. Thus the peripheral structure of G in this case is unique.*

Proof. Under the above assumptions the double $\hat{G}$ of G with respect to the collection of subgroups $\{L_j\}_{j\in J}$ admits a finite Eilenberg-MacLane space $K(\hat{G}, 1)$. Thus we can take as a coarse $PD(n)$-space X the universal cover of $K(\hat{G}, 1)$. Now apply Proposition 8.11. $\square$

9. Applications

In this section we discuss examples of $(n-1)$-dimensional groups which cannot act on coarse $PD(n)$ spaces.

2-dimensional groups with positive Euler characteristic. Let G be a group of type FP_2 with cohomological dimension 2. If the $\chi(G) > 0$ then G cannot act freely simplicially on a coarse $PD(3)$ space. To see this, note that by Mayer-Vietoris some one-ended free factor G' of G must have $\chi(G') > 0$. If G' acts on a coarse $PD(3)$ space then G' contains a collection $\mathcal{H}$ of surface subgroups so that $(G', \mathcal{H})$ is a $PD(3)$ pair. Since the double of a $PD(3)$ pair is a $PD(3)$ group (which has zero Euler characteristic) by Mayer-Vietoris we have $\chi(G') \leq 0$, which is a contradiction.

Bad products. Suppose $G = \prod_{i=1}^k G_i$ where each G_i is a duality group of dimension n_i, and G_1, G_2 are not Poincare duality groups. Then G cannot act freely simplicially on a coarse $PD(n)$ space, where $n-1 = \sum_{i=1}^k n_i$.

Proof. Let $G \curvearrowright X$ be a free simplicial action on a coarse $PD(n)$ space.

Step 1. G contains a $PD(n-1)$ subgroup. This follows by applying Theorem 1.1 to $G \curvearrowright X$, since otherwise $G \curvearrowright X$ is cocompact and Lemma 5.3 would give $H^n(G;\mathbb{Z}G) \simeq \mathbb{Z}$, contradicting $dim(G) = n-1$.

We apply Theorem 1.1 to see that $G \curvearrowright X$ defines deep complementary component stabilizers $H_\alpha \subset G$ which are $PD(n-1)$ groups.

Step 2. Any $PD(n-1)$ subgroup $V \subset G$ virtually splits as a product $\prod_{i=1}^k V_i$ where $V_i \subset G_i$ is a $PD(n_i)$ subgroup. Consequently each G_i contains a $PD(n_i)$ subgroup.

Lemma 9.1. *A $PD(m)$ subgroup V of a m-dimensional product group $W := \prod_{i=1}^k W_i$ contains a finite index subgroup V' which splits as a product $V' = \prod_{i=1}^k V_i$ where $V_i \subset W_i$ is a Poincare duality group of dimension $dim(W_i)$.*

Proof. Look at the kernels of the projections

$$\hat{p}_j : W \to \prod_{i \neq j} W_i$$

restricted to V. The dimension of the middle group in a short exact sequence has dimension at most the sum of the dimensions of the other two groups. Applying this to the exact sequence

$$1 \to W_j \cap V \to V \to \hat{p}_j(V) \to 1$$

we get that $W_j \cap V$ has the same dimension as W_j. Hence $\prod_j (W_j \cap V)$ has the same dimension as V, so it has finite index in V (see section 3). Therefore $\prod_j (W_j \cap V)$ is a $PD(n)$ group and so the factor groups $(W_j \cap V)$ are $PD(dim(W_j))$ groups. $\square$

Step 3. *No $PD(n-1)$ subgroup $V \subset G$ can coarsely separate G.* This follows immediately from step 2 and:

Lemma 9.2. *For $i = 1, 2$ let $A_i \subset B_i$ be finitely generated groups, with $[B_i : A_i] = \infty$. Then $A_1 \times A_2$ does not coarsely separate $B_1 \times B_2$.*

Proof. Suppose that $x = (x_1, x_2), y = (y_1, y_2)$ are points in the Cayley graphs of B_1, B_2 which are at distance at least R from $A := A_1 \times A_2$. Without loss of generality we may assume that $d(x_1, A_1) \geq R/2$. We then pick a point $x_2' \in B_2$ with distance at least $R/2$ from A_2 and connect x_2 to x_2' by a path $x_2(t)$ the the Cayley graph of B_2. The path $(x_1, x_2(t))$ does not intersect $N_{\frac{R}{2}}(A)$. Applying similar argument to y we reduce the proof to the case where $d(x_i, A_i) \geq R/2$ and $d(y_i, A_i) \geq R/2$, $i = 1, 2$. Now connect x_1 to y_1 by a path $x_1(t)$, and y_2 to x_2 by a path $y_2(t)$; it is clear that the paths $(x_1(t), x_2), (y_1, y_2(t))$ do not intersect $N_{\frac{R}{4}}(A)$. On the other hand, these paths connect x to (y_1, x_2) and y to (y_1, x_2). $\square$

Step 4. By steps 1 and 2 we know that each G_i contains a $PD(n_i)$ subgroup. Let $L_i \subset G_i$ be a $PD(n_i)$ subgroup for $i > 1$. Set $L := G_1 \times (\prod_{i=2}^k L_i)$. Observe that L is not a $PD(n-1)$ group since G_1 is not a $PD(n_1)$ group. Therefore no finite index subgroup of L can be a $PD(n-1)$ subgroup, see section 3.

Step 5. Choose a basepoint $\star \in X$. We now apply Theorem 8.8 to the action $L \curvearrowright X$ with $K := L(\star)$, and we let R_i, C_α, H_α E_α, and F_α be as in the Theorem 8.8. Since L has infinite index in G, the distance function $d(\partial C_\alpha, \cdot)$ is unbounded on $G(\star) \cap E_\alpha$ for some $\alpha \in I$, while part 2 of Theorem 8.8 implies that $d(\partial C_\alpha, \cdot)$ is unbounded on $K \cap F_\alpha$. Hence H_α coarsely separates G, which contradicts step 3. $\square$

Baumslag-Solitar groups. Pick $p \neq \pm q$, and let $G := BS(p,q)$ denote the Baumslag-Solitar group with the presentation

$$\langle a, b \mid ba^p b^{-1} = a^q \rangle. \tag{9.3}$$

If G_1 is a k-dimensional duality group then the direct product $G_1 \times G$ does not act freely simplicially on a coarse $PD(3+k)$ space.

We will prove this when $G_1 = \{e\}$. The general case can be proved using straightforward generalization of the argument given below, once one applies the "Bad products" example above to see that G_1 must be a $PD(k)$ group if $G_1 \times G$ acts on a coarse $PD(3+k)$ space. Assume that $G \curvearrowright X$ is a free simplicial action on a coarse $PD(3)$ space. Choosing a basepoint $\star \in X$, we obtain a uniformly proper map $G \to X$.

We recall that the presentation (9.3) defines a graph of groups decomposition of G with one vertex labelled with $\mathbb{Z}$, one oriented edge labelled with $\mathbb{Z}$, and where the initial and final edge monomorphisms embed the edge group as subgroups of index p and q respectively. The Bass-Serre tree T corresponding to this graph of groups has the following structure. The action $G \curvearrowright T$ has one vertex orbit and one edge orbit. For each vertex $v \in T$, the vertex stabilizer G_v is isomorphic to $\mathbb{Z}$. The vertex v has p incoming edges and q outgoing edges; the incoming (respectively outgoing) edges are cyclically permuted by G_v with ineffective kernel the subgroup of index p (respectively q).

Let $\bar{\Sigma}$ be the presentation complex corresponding to the presentation (9.3), and let Σ denote its universal cover. Then Σ admits a natural G-equivariant fibration $\pi : \Sigma \to T$, with fibers homeomorphic to $\mathbb{R}$. For each vertex $v \in T$, the inverse image $\pi^{-1}(v)$ has a cell structure isomorphic to the usual cell structure on $\mathbb{R}$, and G_v acts freely transitively on the vertices. For each edge $e \subset T$, the inverse image $\pi^{-1}(e) \subset \Sigma$ is homeomorphic to a strip. The cell structure on the strip may be obtained as follows. Take the unit square in $\mathbb{R}^2$ with the left edge subdivided into p segments and the right edge subdivided into q segments; then glue the top edge to the bottom edge by translation and take the induced cell structure on the universal cover. The edge stabilizer G_e acts simply transitively on the 2-cells of $\pi^{-1}(e)$.

We may view Σ as a bounded geometry metric simplicial complex by taking a G-invariant triangulation of Σ. Given k distinct ideal boundary points $\xi_1, \dots, \xi_k \in \partial_\infty T$ and a basepoint $\star \in T$, we consider the geodesic rays $\overline{\star\xi_i} \subset T$, take the disjoint union of their inverse images $Y_i := \pi^{-1}(\overline{\star\xi_i}) \subset \Sigma$ and glue them together along the copies of $\pi^{-1}(\star) \subset \pi^{-1}(\overline{\star\xi_i})$. The resulting complex Y inherits bounded geometry metric simplicial complex structure from Σ. The reader will verify the following assertions:

1. Y is uniformly contractible.

2. For $i \neq j$, the union $Y_i \cup Y_j \subset Y$ is uniformly contractible and the inclusion $Y_i \cup Y_j \to Y$ is uniformly proper.

3. The natural map $Y \to \Sigma$ is uniformly proper.

4. The cyclic ordering induced on the Y_i's by the uniformly proper composition $C_*(Y) \to C_*(\Sigma) \to C_*(X)$ (see Lemma 7.11) defines a continous G-invariant cyclic ordering on $\partial_\infty T$.

Let a be the generator of G_v for some $v \in T$. Setting $e_k := (pq)^k$, the sequence $g_k := a^{e_k}$ – viewed as elements in $Isom(T)$ – converges to the identity as $k \to \infty$. So the sequence of induced homeomorphisms of the ideal boundary of T converges to the identity. The invariance of the cyclic ordering clearly implies that g_k acts trivially on the ideal boundary of T for large k. This implies that g_k acts trivially on T for large k. Since this is absurd, G cannot act discretely and simplicially on a coarse $PD(3)$ space.

Remark 9.4. The complex Σ – and hence $BS(p,q)$ – can be uniformly properly embedded in a coarse $PD(3)$ space homeomorphic to $\mathbb{R}^3$. To see this we proceed as follows. First take a proper PL embedding $T \to \mathbb{R}^2$ of the Bass-Serre tree into $\mathbb{R}^2$. For each co-oriented edge $\overrightarrow{e}$ of $T \subset \mathbb{R}^2$ we take product cell structure on the half-slab $P(\overrightarrow{e}) := \pi^{-1}(e) \times \mathbb{R}_+$ where $\mathbb{R}_+$ is given the usual cell structure. We now perform two types of gluings. First, for each co-oriented edge $\overrightarrow{e}$ we glue the half-slab $P(\overrightarrow{e})$ to Σ by identifying $\pi^{-1}(e) \times 0$ with $\pi^{-1}(e) \subset \Sigma$. Now, for each pair $\overrightarrow{e_1}$, $\overrightarrow{e_2}$ of adjacent co-oriented edges, we glue $P(\overrightarrow{e_1})$ to $P(\overrightarrow{e_2})$ along $\pi^{-1}(v) \times \mathbb{R}_+$ where $v = e_1 \cap e_2$. It is easy to see that after suitable subdivision the resulting complex X becomes a bounded geometry, uniformly acyclic 3-dimensional PL manifold homeomorphic to $\mathbb{R}^3$.

Higher genus Baumslag-Solitar groups. Note that $BS(p,q)$ is the fundamental group of the following complex $K = K_1(p,q)$. Take the annulis A with the boundary circles C_1, C_2. Let B be another annulus with the boundary circles C_1', C_2'. Map C_1', C_2' to C_1, C_2 by mappings f_1, f_2 of degrees p and q respectively. Then K is obtained by gluing A and B by $f_1 \sqcup f_2$. Below we describe a "higher genus" generalization of this construction. Instead of the annulus A take a surface S of genus $g \geq 1$ with two boundary circles C_1, C_2. Then repeat the above construction of K by gluing the annulus B to S via the mappings $C_1' \to C_1, C_2' \to C_2$ of the degrees p, q respectively. The fundamental group $G = G_g(p,q)$ of the resulting complex $K_g(p,q)$ has the presentation

$$\langle a_1, b_1, ..., a_g, b_g, c_1, c_2, t : [a_1, b_1]...[a_g, b_g]c_1c_2 = 1, tc_2^q t^{-1} = c_1^p \rangle.$$

One can show that the group $G_g(p,q)$ is torsion-free and Gromov-hyperbolic [27]. Note that the universal cover $\tilde{K}$ of the complex $K_g(p,q)$ does not fiber over the Bass-Serre tree T of the HNN-decomposition of G. Nevertheless there is a properly embedded c_1-invariant subcomplex in $\tilde{K}$ which (c_1-invariantly) fibers over T with the fiber homeomorphic to $\mathbb{R}$. This allows one to repeat the arguments given above for the group $BS(p,q)$ and show that the group $G_g(p,q)$ cannot act simplicially freely on a coarse $PD(3)$ space (unless $p = \pm q$). However in [27] we show that $G_g(p,q)$ contains a finite index subgroup isomorphic to the fundamental group of a compact 3-manifold with boundary.

Groups with too many coarsely non-separating Poincare duality subgroups. By Corollary 7.14, if G is of type FP, and $G \curvearrowright X$ is a free simplicial action on a coarse $PD(n)$ space, then there are only finitely many conjugacy classes of coarsely non-separating maximal $PD(n-1)$ subgroups in G.

We now construct an example of a 2-dimensional group of type FP which has infinitely many conjugacy classes of coarsely non-separating maximal surface subgroups; this example does not fit into any of the classes described above. Let S be a 2-torus with one hole, and let $\{a, b\} \subset H_1(S)$ be a set of generators. Consider a sequence of embedded loops $\gamma_k \subset S$ which represent $a + kb \in H_1(S)$, for $k = 0, 1, \ldots$. Let Σ be a 2-torus with two holes. Glue the boundary torus of $S \times S^1$ homeomorphically to one of the boundary tori of $\Sigma \times S^1$ so that the resulting manifold M is not Seifert fibered. Consider the sequence $T_k \subset M$ of embedded incompressible tori corresponding to $\gamma_k \times S^1 \subset S \times S^1 \subset M$. Let $L \subset \pi_1(M)$ be the infinite cyclic subgroup generated by the homotopy class of γ_0. Finally, we let G be the double of $\pi_1(M)$ over the cyclic subgroup L, i.e. $G := \pi_1(M) *_L \pi_1(M)$. Then the reader may verify the following:

1. Let $H_i \subset \pi_1(M) \subset G$ be the image of the fundamental group of the torus T_i for $i > 0$ (which is well-defined up to conjugacy). Then each H_i is maximal in G, and the H_i's are pairwise non-conjugate in G.

2. Each $H_i \subset \pi_1(M)$ coarsely separates $\pi_1(M)$ into precisely two deep components.

3. For each $i > 0$, the subgroup $H_i \subset \pi_1(M)$ coarsely separates some conjugate of L in $\pi_1(M)$.

4. It follows from 3 that H_i is coarsely non-separating in G for $i > 0$.

5. G is of type FP and has dimension 2.

Therefore G cannot act freely simplicially on a coarse $PD(3)$ space.

10. Appendix: Coarse Alexander duality in brief

We will use terminology and notation from section 2.

Theorem 10.1. *Let X and Y be bounded geometry uniformly acyclic metric simplicial complexes, where X is an n-dimensional PL manifold. Let $f : C_*(Y) \to C_*(X)$ be a uniformly proper chain map, and let $K \subset X$ be the support of $f(C_*(Y)) \subset C_*(X)$. For every R we may compose the Alexander duality isomorphism A.D. with the induced map on compactly supported cohomology:*

$$\tilde{H}_{n-k-1}(X - N_R(K)) \xrightarrow{A.D.} H^k_c(N_R(K)) \xrightarrow{H^k_c(f)} H^k_c(Y); \tag{10.2}$$

we call this composition A_R. Then

1. *For every R there is an R' so that*

$$Ker(A_{R'}) \subset Ker(\tilde{H}_{n-k-1}(X - N_{R'}(K)) \to \tilde{H}_{n-k-1}(X - N_R(K))). \tag{10.3}$$

2. *A_R is an epimorphism for all $R \geq 0$.*

3. *All deep components of $X - K$ are stable, and their number is $1 + rank(H^{n-1}_c(Y))$.*

4. *If Y is an $(n-1)$-dimensional manifold, then for all R there is a D so that any point in $N_R(K)$ lies within distance D of both the deep components of $X - N_R(K)$.*

The functions $R' = R'(R)$ and $D = D(R)$ depend only on the geometry of X and Y (via their dimensions and acyclicity functions), and on the coarse Lipschitz constant and distortion of f.

Proof. Step 1. We construct a coarse Lipschitz chain map $g : C_*(X) \to C_*(Y)$ as follows. For each vertex $x \in X, y \in Y$ we let $[x], [y]$ denote the corresponding element of $C_0(X), C_0(Y)$. To define $g_0 : C_0(X) \to C_0(Y)$ we map $[x]$ for each vertex $x \in X \subset C_0(X)$ to $[y]$, where we choose a vertex $y \in Y \subset C_0(Y)$ for which the distance $d(x, Support(f(y)))$ is minimal, and extend this homomorphism $\mathbb{Z}$-linearly to a map $C_0(X) \to C_0(Y)$. Now assume inductively that $g_j : C_j(X) \to C_j(Y)$ has been defined by $j < i$. For each i-simplex $\sigma \in C_i(X)$, we define $g_i(\sigma)$ to be a chain bounded by $g_{i-1}(\partial\sigma)$ (where $Support(g_i(\sigma))$ lies inside the ball supplied by the acyclicity function of Y). Using a similar inductive procedure to construct chain homotopies, one verifies:

a) For every R there is an R' so that the composition

$$C_*(N_R(K)) \xrightarrow{g_*} C_*(Y) \to C_*(K) \to C_*(N_{R'}(K)) \tag{10.4}$$

is chain homotopic to the inclusion by an R'-Lipschitz chain homotopy with displacement $< R'$.

b) There is a D so that

$$C_*(Y) \xrightarrow{f} C_*(K) \xrightarrow{g} C_*(Y)$$

is a chain map with displacement at most D and $g \circ f$ is chain homotopic to $id_{C_*(Y)}$ by a D-Lipschitz chain map with displacement $< D$.

Step 2. Pick R, and let R' be as in a) above. If

$$\alpha \in Ker(H_c^k(N_{R'}(K)) \xrightarrow{H_c^k(f)} H_c^k(Y)),$$

then α is in the kernel of the composition

$$H_c^k(N_{R'}(K)) \xrightarrow{H_c^k(f)} H_c^k(Y) \xrightarrow{H_c^k(g)} H_c^k(N_R(K))$$

which coincides with the restriction $H_c^k(N_{R'}(K)) \to H_c^k(N_R(K))$ by a) above. Similarly, the composition

$$H_c^k(Y) \xrightarrow{H_c^k(g)} H_c^k(N_R(K)) \xrightarrow{H_c^k(f)} H_c^k(Y)$$

is the identity, so $H_c^k(f)$ is an epimorphism. Applying the Alexander duality isomorphism to these two assertions we get parts 1 and 2.

Step 3. Let C be a deep component of $X - K$. Suppose C_1, C_2 are deep components of $X - N_R(K)$ with $C_i \subset C$. Picking points $x_i \in C_i$, the difference $[x_1] - [x_2]$ determines an element of $\tilde{H}_0(X - N_R(K))$ lying in $Ker(\tilde{H}_0(X - N_R(K)) \to \tilde{H}_0(X - K)$. Hence

$$A_R([x_1] - [x_2]) = A_0(p_R([x_1] - [x_2])) = A_0(0) = 0$$

where $p_R : \tilde{H}_0(X - N_R(K)) \to \tilde{H}_0(X - K)$ is the projection. Since C_1 and C_2 are deep, for any $R' \geq R$ there is a $c \in \tilde{H}_0(X - N_{R'}(K))$ which projects to $[x_1] - [x_2] \in \tilde{H}_0(X - N_R(K))$. But then $A_{R'}(c) = 0$ and part 1 forces $[x_1] - [x_2] = 0$. This proves that $C_1 = C_2$, and hence that all deep components of $X - K$ are stable. The number of deep components of $X - K$ is

$$1 + rank(\lim_{\overleftarrow{R}} \tilde{H}_0(X - N_R(K)),$$

and by part 1 this clearly coincides with $1 + rank(H_c^{n-1}(Y))$. Thus we have proved 2.

Step 4. To prove part 4, we let C_1, C_2 be the two deep components of $X - K$ guaranteed to exist by part 3. Pick $x \in N_R(K)$, and let R' be as in part 1. Since f is coarse Lipschitz chain map, there is a $y \in Y$ with $d(x, Support(f([y]))) < D_1$ where D_1 is independent of x (but does depend on R). Choose a cocycle $\alpha \in C_c^{n-1}(Y)$ representing the generator of $H_c^{n-1}(Y)$ which is supported in an $(n-1)$-simplex containing y. Then the image α' of α under $C_c^{n-1}(Y) \stackrel{C_c^{n-1}(g)}{\longrightarrow} C_c^{n-1}(N_{R'}(K))$ is a cocycle supported in $B(x, D_2) \cap N_{R'}(K)$ where D_2 depends on R' but is independent of x. Applying the Alexander duality isomorphism[11] to $[\alpha'] \in H_c^{n-1}(N_{R'}(K))$, we get an element $c \in \tilde{C}_0(X - N_{R'}(K))$ which is supported in $B(x, D_2 + 1) \cap (X - N_{R'}(K))$, and which maps under $A_{R'}$ to $[\alpha] \in H_c^{n-1}(Y)$. Picking $x_i \in C_i$ far from K, we have $[x_1] - [x_2] \in \tilde{H}_0(X - N_{R'}(K))$ and $A_{R'}([x_1] - [x_2]) = \pm[\alpha]$. By part 1 it follows that the images of c and $[x_1] - [x_2]$ under the map $\tilde{H}_0(X - N_{R'}(K)) \to \tilde{H}_0(X - N_R(K))$ coincide up to sign. In other words, $support(c) \cap C_i \neq \emptyset$, so we've shown that $d(x, C_i) < D_2$ for each $i = 1, 2$. $\square$

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Michael Kapovich:

Department of Mathematics
University of Utah
Salt Lake City, UT 84112-0090
kapovich@math.utah.edu

Bruce Kleiner:

Department of Mathematics
University of Michigan
Ann Arbor, MI 48109
bkleiner@math.lsa.umich.edu