Every Compact, Real Algebraic Variety is Metrically Equivalent to a Disjoint Part of the Configuration Space of a Mechanical Linkage

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Abstract

We present a constructive method to represent every compact, real algebraic variety as a metrically equivalent disjoint part of the configuration space of a mechanical linkage in the Euclidean plane $\mathbb{R}^2$. The aim is to construct addition and multiplication with weighted graphs in $\mathbb{R}^2$ that permit a "mechanical realization" of polynomial functions, and thus of varieties.

1 Introduction

Calculating the configuration space of several weighted graphs, we found that all compact, closed, oriented two-manifolds arise as the configuration space of a simple mechanical linkage, [4] and [6]. We could not find any mechanical linkage with a nonorientable configuration space. However, in [5] the following statement can be found: *Toute courbe algébrique peut être tracée à l'aide d'un système articulé.* It is a known fact that the configuration space of connected weighted graphs in the Euclidean plane are compact, real algebraic varieties naturally embedded in $(\mathbb{R}^2)^m$, where m is the number of vertices in the graph. It is only natural to ask whether every compact, real algebraic variety arise as the configuration space of some mechanical linkage in the Euclidean plane.

2 Preliminaries

2.1 Definitions

The first step is to replace our intuitive imagined idea of a mechanical linkage by an exact mathematical definition:

Definition 2.1 The tripel $\mathcal{G} = (V, E, d)$ consisting of

(1) a set of *vertices* $V = \{V_1, \ldots, V_m\}$,

(2) a set of *edges* $E = \{\{V_{i_1}, V_{j_1}\}, \{V_{i_2}, V_{j_2}\}, \ldots, \{V_{i_k}, V_{j_k}\}\}$ with $i_l, j_l \in \{1, \ldots, m\}$, $i_l \neq j_l$ and

(3) a *weight* $d : E \to \mathbb{R}_+^*$, that attaches to every edge $\{V_{i_l}, V_{j_l}\}$ a length (weight) $d(V_{i_l}, V_{j_l}) \in \mathbb{R}_+^*$,

is called a *weighted graph.*

Definition 2.2 A weighted graph $\mathcal{G} = (V, E, d)$ is called *realisable* in $\mathbb{R}^2$, if a mapping

$$\phi : V \to \mathbb{R}^2$$

exists, such that $\phi(V_j) \in \mathbb{R}^2$ with $d_{\mathbb{R}}(\phi(V_i), \phi(V_j)) = d(V_i, V_j)$ for all $\{V_i, V_j\} \in E$, where $d_{\mathbb{R}}$ is the Euclidean metric in $\mathbb{R}^2$.

We identify V_j with $\phi(V_j)$, and since the following refers only to realisable graphs $\mathcal{G}$ in $\mathbb{R}^2$, we speak of *weighted graphs* in $\mathbb{R}^2$ or of *mechanical linkages* only.

Definition 2.3 A *realization* ξ of $\mathcal{G} = (V, E, d)$ is a set $\{V_1, \ldots, V_m\}$ of points in $\mathbb{R}^2$ such that the weight d is respected by all edges $\{V_i, V_j\} \in E$.

According to these definitions every realization of a mechanical linkage in $\mathbb{R}^2$ can be represented by a drawing consisting of a set of points V and a set of straight lines E with the prescribed length (weight) d.
The above definition makes sense, as there are simple graphs $\mathcal{G}$ with three vertices that are not realisable in the Euclidean plane. Obviously such a realization is in general not unique.

The configuration space $[\mathcal{G}]$ of a weighted graph $\mathcal{G} = (V, E, d)$ in $\mathbb{R}^2$ is defined as a metric subspace of $(\mathbb{R}^2)^m$.

Definition 2.4 Let $\{V_{i_0}, V_{j_0}\} \in E$ be any edge of $\mathcal{G}$ fixed in $\mathbb{R}^2$. The configuration space of $\mathcal{G}$ is defined by

$$\begin{aligned}[\mathcal{G}] \ := \ & \{(P_1, \dots, P_m) \in (\mathbb{R}^2)^m; \\ & P_{i_0} = V_{i_0}, P_{j_0} = V_{j_0},\ i_0 \neq j_0, d_{\mathbb{R}}(P_i, P_j) = d(V_i, V_j) \\ & \forall \{V_i, V_j\} \in E\}\end{aligned}$$

with the induced metric of $\mathbb{R}^{2m}$.

Remark 2.5 The configuration space $[\mathcal{G}]$ is a topological space with the induced topology of the metric of $[\mathcal{G}]$.

Often we are only interested in subsets of the configuration space $[\mathcal{G}]$ of a given mechanical linkage $\mathcal{G} = (V, E, d)$. Fixing an edge of $\mathcal{G}$ and restricting a number of vertices in E on a subset $X \subset \mathbb{R}^2$, allows us to consider one subset of $[\mathcal{G}]$ only:

$$\{\xi \in [\mathcal{G}]; V_j \in X \subset \mathbb{R}^2, j \in A \subset \{1, \dots, m\}\}.$$

In all figures where a mechanical linkage $\mathcal{G}$ is represented, only one realization of $[\mathcal{G}]$ is drawn. The other realizations are represented as follows: An empty small circle around a vertex indicates a vertex fixed in the plane; a filled circle around a vertex indicates a joint, and a filled halfcircle between two vertices indicates a joint with another edge, the vertex of which lies on the edge with which the filled halfcircle connects. This and the remark above enables us to speak of movements of vertices and of rigid mechanisms:

Definition 2.6 Let $\mathcal{G}$ be any mechanical linkage in $\mathbb{R}^2$ with a fixed edge. A vertex V_i of $\mathcal{G}$ can be *moved* on $A \subset \mathbb{R}^2$ with respect to this edge, if a realization of $\mathcal{G}$ in the same component of $[\mathcal{G}]$ with $\phi(V_i) = P$ exists for all points $P \in A$. V_i is called *rigid* with respect to the fixed edge and a realization $\xi \in [\mathcal{G}]$, if a neighbourhood $U \subset \mathbb{R}^2$ of P exists for all $P := \phi(V_i)$ such that for all realizations with $\phi(V_i) \in U$ follows $\phi(V_i) = P$.

Definition 2.7 A realization $\xi \in [\mathcal{G}]$ of a mechanical linkage $\mathcal{G}$ is called *rigid* in $\mathbb{R}^2$, if every vertex of $\mathcal{G}$ is rigid with respect to the fixed edge and a realization ξ.

If a realization is rigid with respect to one fixed edge, then it is also rigid with respect to any other, and $[\mathcal{G}] \approx pt \coprod T_{rest}$, where T_{rest} is any topological space.

2.2 Real Algebraic Geometry

First we recall some facts about *real* algebraic geometry, that serve as a basis for our considerations.

Definition 2.8 Let A be a subset of $\mathbb{R}[X_1,\ldots,X_n]$. The set

$$\mathcal{L}(A) := \{x \in \mathbb{R}^n; \forall f \in A \;\; f(x) = 0\}$$

is called the *set of zeros* of A.
Let S be a subset of $\mathbb{R}^n$. We note

$$\mathcal{F}(S) := \{f \in \mathbb{R}[X_1,\ldots,X_n]; \forall x \in S \;\; f(x) = 0\}.$$

The *real algebraic varieties* are those subsets V of $\mathbb{R}^n$, for which

$$\mathcal{L}(\mathcal{F}(V)) = V.$$

The ring $\mathbb{R}$ is Noetherian, and we can use *Hilbert's Basis Theorem* to show that every ideal in $\mathbb{R}[X_1,\ldots,X_n]$ is finitely generated. Since every evaluation of a polynomial in A on $\mathbb{R}^n$ is real, V in $\mathbb{R}^n$ can always be described as the zero set of a single polynomial $F \in \mathbb{R}[X_1,\ldots,X_n]$.

Lemma 2.9 *For each real algebraic variety $V \subset \mathbb{R}^n$ a polynomial $F \in \mathbb{R}[X_1,\ldots,X_n]$ exists such that $V = \mathcal{L}(F)$.*

PROOF LEMMA We define $F := f_1^2 + \ldots + f_l^2$, where $f_1,\ldots,f_l$ are the generators of the ideal $\mathcal{F}(V)$. q.e.d

2.3 Special Mechanical Linkages in $\mathbb{R}^2$

The proof of the main theorem builds on knowledge about special mechanical linkages, i.e. the following three basic mechanisms, cf. [5].

2.3.1 Translator $\mathcal{T}$

We look at the mechanism $\mathcal{T}$ in figure 1.
Let O be fixed in $\mathbb{R}^2$ with $d(O,A) = d(B,E) = d(C,D)$, $d(O,E) = d(A,B)$ and $d(E,D) = d(B,C)$ in $\mathbb{R}_+^*$. The vertices $OABE$ and $EBCD$ form a parallelogram for all realizations of $\mathcal{T}$. If these prerequisits are valid for one realization, then they are for all, and the following properties for $\mathcal{T}$ are true:

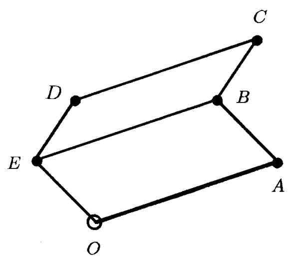

Figure 1: Translator $\mathcal{T}$

(a) $\mathcal{T}$ translates the edge $\{O, A\}$ into the vertex D.

(b) $\mathcal{T}$ translates the straight line OD into the vertex A.

(c) $\{\xi \in [\mathcal{T}]; D \neq O, \;\; D, A \in \mathbb{R}^2 \text{ fixed}\} \approx pt \coprod pt$.

(d) $\{\xi \in [\mathcal{T}]; D = O, \;\; A \in \mathbb{R}^2 \text{ fixed}\} \approx S^1$.

Further on we use $\mathcal{T}$ such that there occurs no realization $\xi \in [\mathcal{T}]$ with $D = O$. This can be attained either with $d(O, E) \neq d(E, D)$ or a suitable size of $\mathcal{T}$ and the coupling of $\mathcal{T}$ onto other mechanisms.

2.3.2 Rotator-Dilatator $\mathcal{R}$

We consider the mechanism $\mathcal{R}$ in figure 2.

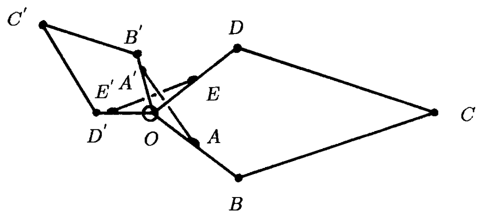

Figure 2: Rotator-Dilatator $\mathcal{R}$

Let O be fixed in $\mathbb{R}^2$ with $d(O, B) = d(O, D)$ and $d(B, C) = d(C, D)$ in $\mathbb{R}_+^*$. The triangles OAA' and OEE' are congruent, and the kite quadrilaterals $OBCD$ and $OB'C'D'$ are similar. If these statements are true for one realization, then they are for all, and we obtain the following features of $\mathcal{R}$:

(a) The straight line OC is turned and dilatated in OC'.

(b) $\{\xi \in [\mathcal{R}]; C \neq O,\ \ C \in \mathbb{R}^2 \text{ fixed}\} \approx pt \coprod \ldots \coprod pt.$

(c) $\{\xi \in [\mathcal{R}]; C = O\} \not\approx pt \coprod \ldots \coprod pt.$

Further on we use $\mathcal{R}$ such that for each realization we never have $C = O$.

2.3.3 Sextilataire of Kempe $\mathcal{D}$

We consider the mechanism $\mathcal{D}$ in figure 3.

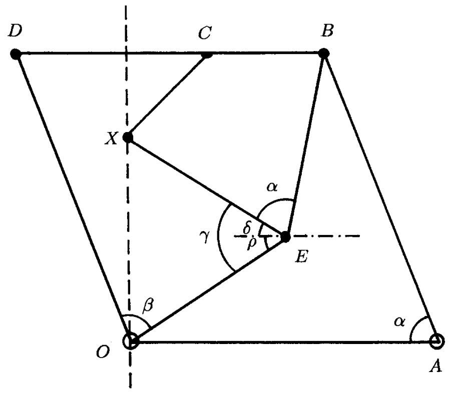

Figure 3: Sextilataire of Kempe $\mathcal{D}$

Let O and A be fixed in $\mathbb{R}^2$ and the following conditions be met: $d(O,E) < d(O,A)$, $OABD$ is a rhombus, $OEBD$ and $CXEB$ are similar kite quadrilaterals. These conditions are true for all realizations of $\mathcal{D}$.
Our interest lies in a subset of the configuration space of $\mathcal{D}$. In order to be able to deal with the subject we need some definitions and lemmas: In the point O we put a local orthogonal system of coordinate of $\mathbb{C}$, such that O lies in the origin of $\mathbb{C}$ and A on the positive real axis.

Lemma 2.10 *If we define for any $\varepsilon \in \,]0,1[\; I_\varepsilon := \{z \in \mathbb{C}; \varepsilon \leq Im(z) \leq 2 \cdot d(O,E) - \varepsilon,\ Re(z) = 0\}$, then an open neighbourhood U_{I_ε} in $\mathbb{C}$ of I_ε exists such that from $X \in U_{I_\varepsilon}$ follows that $X \in I_\varepsilon$.*

PROOF LEMMA We define the angles $\alpha := \measuredangle(OAB)$ and $\beta := \measuredangle(EOD)$. Also $\delta = \pi - \alpha - \beta$ and

$$\gamma \;=\; \rho - \delta$$

$$\begin{aligned} &= ((2\pi - 2\beta - \alpha) - \alpha) - (\pi - \beta - \alpha) \\ &= \pi - \beta - \alpha \\ &= \delta \end{aligned}$$

Since $d(O,E) = d(E,X)$, there is $OX \perp OA$, so $X \in I_\varepsilon$. q.e.d

Lemma 2.11 *X can move on one part of the imaginary axis namely on*

$$I_0 := \{z \in \mathbb{C}; 0 \le Im(z) \le 2d(O,E), Re(z) = 0\} \supset I_\varepsilon.$$

I.e., X can move on a straight line.

PROOF LEMMA We give a parametrisation of X:

$$\begin{aligned} X &= d(O,E) \cdot e^{i(\pi-\beta-\alpha)} + d(E,X) \cdot e^{i(\pi-\gamma)} \\ &= d(O,E) \cdot (-e^{-i(\beta+\alpha)} + e^{i(\beta+\alpha)}) \\ &= 2i \cdot d(O,E) \cdot \sin(\beta+\alpha) \end{aligned}$$

This parametrisation is true for all $\beta + \alpha \in]\frac{\pi}{2}, \pi[$. q.e.d

We define $I_{max} := \{z \in \mathbb{C}; 0 < Im(z) < 2 \cdot d(O,E), Re(z) = 0\}$ and obtain

Lemma 2.12 *Let $X \in I_{max}$ be fixed. Then $\xi \in [\mathcal{D}]$ is rigid in $\mathbb{R}^2$.*

PROOF LEMMA Let X be fixed in $\mathbb{R}^2$. It follows that E is rigid in $\mathbb{R}^2$, since O is fixed in $\mathbb{R}^2$; B is also rigid in $\mathbb{R}^2$, since A is fixed in $\mathbb{R}^2$; C and D are also rigid in $\mathbb{R}^2$, because O is fixed in $\mathbb{R}^2$. Therefore the considered realization is rigid. q.e.d

Summarizing the above we obtain two important corollaries:

Corollary 2.13 $\{\xi \in [\mathcal{D}]; X \in I_{max}\} = I_{max} \coprod T_{rest}$, *where T_{rest} is any topological space. q.e.d*

Corollary 2.14 *For each $\varepsilon \in]0,1[$ a neighbourhood U_{I_ε} of I_ε exists such that $\{\xi \in [\mathcal{D}]; X \in U_{I_\varepsilon}\} = I_\varepsilon \coprod T_{rest}$, where T_{rest} is any topological space. q.e.d*

2.4 Product Mechanism

Definition 2.15 Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two mechanical linkages in $\mathbb{R}^2$. Then the product mechanism $\mathcal{G}_1 \otimes \mathcal{G}_2$ is defined by fixing any edge of each $\mathcal{G}_1$ and $\mathcal{G}_2$ anywhere in $\mathbb{R}^2$.

We get the following

Lemma 2.16 $[\mathcal{G}_1 \otimes \mathcal{G}_2] = [\mathcal{G}_1] \times [\mathcal{G}_2]$.

Proof Lemma $\mathcal{G}_1$ and $\mathcal{G}_2$ create independently the configuration spaces $[\mathcal{G}_1]$ and $[\mathcal{G}_2]$. q.e.d

In the definition of $\mathcal{G}_1 \otimes \mathcal{G}_2$ any edge of $\mathcal{G}_1$ and $\mathcal{G}_2$ can be fixed in $\mathbb{R}^2$. Later on we couple mechanical linkages. Thus the geometry of the whole mechanism, and the choice of the fixed edges are relevant for the configuration space. If we proceed according to definition 2.6, then the fixing can be done by introducing edges: that results a rigid connection.

3 Universality for Compact, Real Algebraic Varieties

Theorem 3.1 *Let $V \subset \mathbb{R}^n$ be a compact, real algebraic variety. Then V is metrically identic to a disjoint part T_j of the configuration space $[\mathcal{H}] = \coprod_{j=1}^{k} T_j$ of a mechanical linkage $\mathcal{H}$ in $\mathbb{R}^2$. The variety V has the induced metric of $\mathbb{R}^n$.*

Proof Theorem

1. We are interested in compact, real algebraic varieties in $\mathbb{R}^n$ that occur as the zero set of a single polynomial $F \in \mathbb{R}[X_1, \ldots, X_n]$.

   We consider the canonical representation of $F \in \mathbb{R}[X_1, \ldots, X_n]$ as a finite real linear combination of the monoms $X_1^{\nu_1} \cdots X_n^{\nu_n} \in \mathbb{R}[X_1, \ldots, X_n]$:

   $$F = \sum_{(\nu_1, \ldots, \nu_n) \in \mathbb{N}_0^n} a_{\nu_1, \ldots, \nu_n} X_1^{\nu_1} \cdots X_n^{\nu_n}$$

   with $a_{\nu_1, \ldots, \nu_n} \in \mathbb{R}$, most of all are zero.

F is constructed algorithmically from the zero polynomial with addition, scalar multiplication and multiplication of the terms $a_{\nu_1,\ldots,\nu_n}$ und $X_1, \ldots, X_n$:
We define the *lexicographic order* $\geq$ on $\mathbb{N}_0^n$ and get $(\mathbb{N}_0^n, \geq)$.

Definition 3.2 *Let $(\nu_1, \ldots, \nu_n)$ and $(\nu_1', \ldots, \nu_n')$ be in $\mathbb{N}_0^n$. It follows that*

$$(\nu_1, \ldots, \nu_n) \geq (\nu_1', \ldots, \nu_n')$$

if and only if $\nu_1 > \nu_1'$ or $(\nu_1 = \nu_1'$ and $\nu_2 > \nu_2')$ or $\ldots$ or $(\nu_1 = \nu_1', \ldots, \nu_n = \nu_n')$.

Constructing F we start with the smallest element $(0, \ldots, 0) \in \mathbb{N}_0^n$ and define the set $\mathcal{U}_s(F)$ of special subpolynomials of F, consisting of the following elements:

$$u_{(0,\ldots,0)}^{(0,\ldots,0)} := a_{0,\ldots,0}$$

The result is an order of construction for the summands of F.

$$\begin{aligned} u_{(1,0,\ldots,0)}^{(0,\ldots,0)} &:= a_{1,0,\ldots,0} \\ u_{(1,0,\ldots,0)}^{(1,0,\ldots,0)} &:= a_{1,0,\ldots,0} X_1 \end{aligned}$$

The construction proceeds in this manner up to the biggest multiindex $(\hat{\nu}_1, \ldots, \hat{\nu}_n)$ that appears in F. As a result we get the monom of maximal total degree in F.

$$\begin{aligned} u_{(\hat{\nu}_1,\ldots,\hat{\nu}_n)}^{(0,\ldots,0)} &:= a_{\hat{\nu}_1,\ldots,\hat{\nu}_n} \\ &\vdots \\ u_{(\hat{\nu}_1,\ldots,\hat{\nu}_n)}^{(\iota_1,\ldots,\iota_n)} &:= a_{\hat{\nu}_1,\ldots,\hat{\nu}_n} X_1^{\iota_1} \cdots X_n^{\iota_n} \\ &\vdots \\ u_{(\hat{\nu}_1,\ldots,\hat{\nu}_n)}^{(\hat{\nu}_1,\ldots,\hat{\nu}_n)} &:= a_{\hat{\nu}_1,\ldots,\hat{\nu}_n} X_1^{\hat{\nu}_1} \cdots X_n^{\hat{\nu}_n} \end{aligned}$$

Further we define the set $\mathcal{U}_c(F)$

$$\{v^{(\nu_1,\ldots,\nu_n)} := \sum_{(\iota_1,\ldots,\iota_n) \leq (\nu_1,\ldots,\nu_n)} u_{(\iota_1,\ldots,\iota_n)}^{(\iota_1,\ldots,\iota_n)};\ (\nu_1, \ldots, \nu_n) \leq (\hat{\nu}_1, \ldots, \hat{\nu}_n)\}$$

of composed subpolynomials of F and the union of all special and composed subpolynomials

$$\mathcal{U}(F) := \mathcal{U}_s(F) \cup \mathcal{U}_c(F).$$

Example 3.3 F is any polynomial in two variables, F does not need to have a compact zero set.

$$\begin{aligned} F(X,Y) &:= 5X^2 - 3XY + 2 \\ \mathcal{U}_s(F) &= \{2, -3, -3X, -3XY, 5, 5X, 5X^2\} \\ \mathcal{U}_c(F) &= \{2, 2-3XY, 2-3XY+5X^2\} \end{aligned}$$

We define the real numbers

$$\begin{aligned} A_F &:= \min\{u(x); \forall x \in V, \forall u \in \mathcal{U}(F)\}, \\ B_F &:= \max\{u(x); \forall x \in V, \forall u \in \mathcal{U}(F)\}, \\ C_F &:= \max\{|x_j|; \forall x \in V, \forall j \in \{1, \ldots, n\}\} \end{aligned}$$

and from that

$$L_F := 1 + \max\{|A_F|, |B_F|, C_F\}.$$

Since $\mathcal{U}(F) \subset \mathbb{R}[X_1, \ldots, X_n]$ and V is compact, these extrema exist.

2. With the product $\mathcal{D}_1 \otimes \cdots \otimes \mathcal{D}_n$ of n identical Sextilataires of Kempe $\mathcal{D}$ a system of coordinates on $]-L_F, L_F[^n$ is constructed:

   For this purpose we set the length $d(O_j, E_j) := L_F$ in $\mathcal{D}_j$ for all $j \in \{1, \ldots, n\}$ and obtain with the vertex X_j all points in $]-L_F, L_F[$. This interval is identified with I_{max}. Identifying the edges $\{O_j, A_j\}$ of all n mechanisms $\mathcal{D}_j$, we get

   $$[\mathcal{D}_1 \otimes \cdots \otimes \mathcal{D}_n] = [\mathcal{D}_1] \times \cdots \times [\mathcal{D}_n].$$

   The restriction of the vertices $X_1, \ldots, X_n$ on $]-L_F, L_F[$ produces the restricted configuration space

   $$\{\xi \in [\mathcal{D}_1 \otimes \cdots \otimes \mathcal{D}_n];\, X_1, \ldots, X_n \in \,]-L_F, L_F[\} = \,]-L_F, L_F[^n \amalg T_{rest}.$$

   $V = \mathcal{L}(F)$ is a compact, real algebraic variety in $\mathbb{R}^n$, and the evaluation of all the subpolynomials $u \in \mathcal{U}(F)$ on V is in $]-L_F, L_F[$. Our next aim is to couple the known special mechanical linkages, so that they define an addition and a multiplication in $]-L_F, L_F[$.

3. Realization of the additioner $\mathcal{A}$ and the multiplicator $\mathcal{M}$:

   Let x and y be the evaluations of two subpolynomials $u, v \in \mathcal{U}(F)$ at the point $x_0 \in V$, i.e. $u(x_0) = x$ and $v(x_0) = y$. With construction 1. we assume that x, y, $x+y$, $x \cdot y$, $a+x$, $a \cdot x \in \,]-L_F, L_F[$, where $a \in \,]-L_F, L_F[$ is any coefficient of F. For each mechanism $\mathcal{A}$ and $\mathcal{M}$, the values of x and y as the evaluations $x+y$ of $\mathcal{A}$ and $x \cdot y$ of $\mathcal{M}$ are attached to the corresponding vertices $X, Y, X+Y, X \cdot Y$. Then $\mathcal{A}$ and $\mathcal{M}$ need to have the following properties:

   - $\mathcal{A}$ and $\mathcal{M}$ can be placed in such a way that X and Y move on the straight line $]-L_F, L_F[$.
   - If X and Y are in $]-L_F, L_F[$, then $X+Y$ are in $]-L_F, L_F[$ for $\mathcal{A}$ and $X \cdot Y$ are in $]-L_F, L_F[$ for $\mathcal{M}$.
   - The realization of the addition and multiplication with $\mathcal{A}$ and $\mathcal{M}$ on $]-L_F, L_F[$ is well defined.
   - If $\xi \in [\mathcal{A}]$ and two vertices of $\{X, Y, X+Y\}$ in $\mathcal{A}$ are fixed in $\mathbb{R}^2$, then $\xi \in [\mathcal{A}]$ is rigid.
     If $\xi \in [\mathcal{M}]$ and two vertices of $\{X, Y, X \cdot Y\}$ in $\mathcal{M}$ are fixed in $\mathbb{R}^2$, then $\xi \in [\mathcal{M}]$ is rigid.

4. Additioner $\mathcal{A}$:

   $\mathcal{A}$ consists of three translators $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3$ as shown in figure 4.

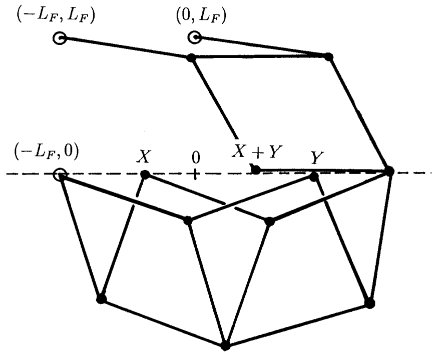

Figure 4: Additioner $\mathcal{A}$

   Respecting the properties of 3., we take the following scaling for all $\mathcal{T}_j$:

$d(O, A) = d(O, E) = d(D, E) := L_F.$

The mechanism $\mathcal{A}_a$ for the addition with a constant $a \in \,]-L_F, L_F[$ is gained from $\mathcal{A}$, substituting $Y := a$.

5. Multiplicator $\mathcal{M}$:

   The realization of the multiplication with $\mathcal{M}$ is based on a construction with a pair of compasses and a ruler, cf. figure 5.

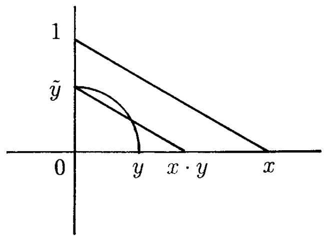

Figure 5: ratio $x : 1 = (x \cdot y) : y$

$\mathcal{M}$ is constructed by coupling two submechanisms:

The first is used to rotate the vertex Y to $\tilde{Y}$ around the origin by $\frac{\pi}{2}$ and is composed as figure 6 shows. It is a coupling of a rotator-dilatator $\mathcal{R}$ with dilatation factor one and two translators $\mathcal{T}_1, \mathcal{T}_2$.

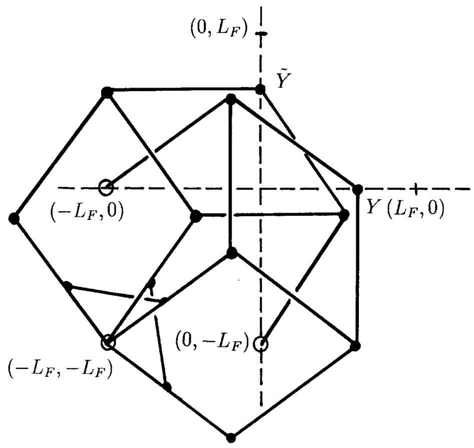

Figure 6: Rotator $\mathcal{R}$

Respecting the properties of 3., we take the following scaling for $\mathcal{T}_1, \mathcal{T}_2$:
$d(O, A) = d(O, E) = d(D, E) := L_F$. $\mathcal{R}$ is exactly determined by $\mathcal{T}_1, \mathcal{T}_2$.

The second submechanism of $\mathcal{M}$ is responsable for the rest of the construction. It consists of three translators $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3$, three Sextilataires of Kempe $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3$, and two edges K_1 and K_2 for the orthogonal fixing of $\mathcal{D}_1$ and $\mathcal{D}_2$, cf. figure 7.

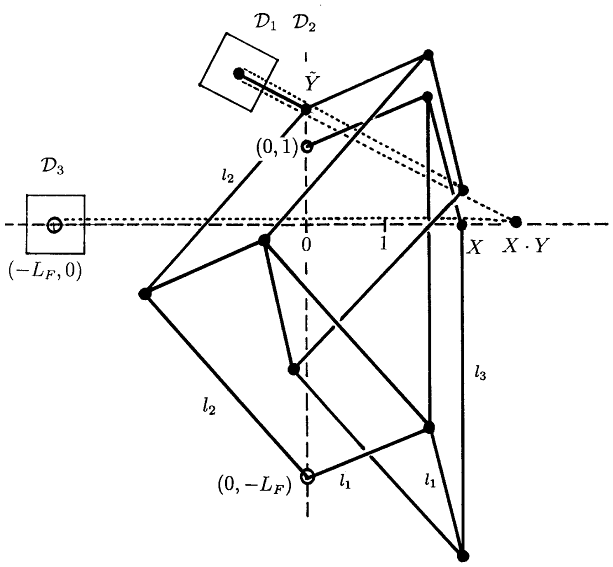

Figure 7: Multiplicator

Respecting the properties of 3., we take the following scaling: $l_1 := \frac{1}{2}\sqrt{1 + L_F^2}$, $l_2 := L_F$ and $l_3 := 1 + L_F$. Thus the three translators $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3$ are laid down, and $\mathcal{D}_3$ is determined by the length $d(O, A) := L_F$. The length of the edge K_1 is one and is used to fix the vertex O in $\mathcal{D}_1$ and $\mathcal{D}_2$ such that $O \neq \tilde{y}$. With the edge K_2 we fix $\{O_1, A_1\}$ of $\mathcal{D}_1$ and $\{O_2, A_2\}$ of $\mathcal{D}_2$ orthogonally on K_1. $\mathcal{D}_1$ and $\mathcal{D}_2$ are scaled by $d(O, A) := \frac{1}{2}(1 + \sqrt{2}L_F)$. $\mathcal{D}_1$ gives the direction of the line going through $\tilde{Y}$ and $X \cdot Y$. $\mathcal{D}_2$ and $\mathcal{D}_3$ form the cutting point $X \cdot Y$ on the straight line $]-L_F, L_F[$.

The mechanism $\mathcal{M}_a$ for the scalar multiplication is gained from

$\mathcal{M}$, substituting $Y := a$.

6. Generating a mechanism $\mathcal{H}$ by coupling mechanisms $\mathcal{D}_1 \otimes \cdots \otimes \mathcal{D}_n$, $\mathcal{A}$, $\mathcal{A}_a$, $\mathcal{M}$ and $\mathcal{M}_a$ to represent the polynomial F with $V = \mathcal{L}(F)$:

   Starting from the variables $x = (x_1, \ldots, x_n) \in \,]-L_F, L_F[^n$ the polynomial F is constructed as a mechanism $\tilde{\mathcal{H}}$. The described algorithmic procedure to build up F with mechanisms $\mathcal{A}$, $\mathcal{A}_a$, $\mathcal{M}$ and $\mathcal{M}_a$ is sufficant for that the polynomial F is represented by $\tilde{\mathcal{H}}$ only in a neighbourhood $U(V) \subset \,]-L_F, L_F[^n$ of V.
   All subpolynomials $u \in \mathcal{U}(F)$ are restricted, in particular $u(x) \in \,]-L_F, L_F[\ \forall x \in V$ and because of the scaling and the construction the final mechanism $\tilde{\mathcal{H}}$ is well defined in $U(V)$, i.e. for every realization $\xi \in [\tilde{\mathcal{H}}]$ with $(X_1, \ldots, X_n) \in U(V)$ fixed, follows $\xi \in [\tilde{\mathcal{H}}]$ is rigid. Thus

   $$\{\xi \in \tilde{\mathcal{H}}\,;(X_1, \ldots, X_n) \in V\} = V \amalg T_{rest}.$$

   Let the vertices $X_1, \ldots, X_n$ be restricted on V. Since

   $$V = \{(x_1, \ldots, x_n) \in \mathbb{R}^n;\ F(x_1, \ldots, x_n) = 0\},$$

   the vertex $F(X_1, \ldots, X_n)$ of $\tilde{\mathcal{H}}$, that represents the evaluation of F is always in the origin of the interval $]-L_F, L_F[$. If $(X_1, \ldots, X_n) \in U(V) - V$, then the vertex $F(X_1, \ldots, X_n)$ of $\tilde{\mathcal{H}}$ is never in zero. These considerations result in the

   Corollary 3.4 *If the vertex $F(X_1, \ldots, X_n)$ is fixed in $0 \in \,]-L_F, L_F[$, then either*

   (i) $(X_1, \ldots, X_n) \in V \subset \,]-L_F, L_F[^n$ *or*

   (ii) $(X_1, \ldots, X_n) \notin U(V)$. *q.e.d*

   If the vertex $F(X_1, \ldots, X_n)$ is fixed in zero, and if it is true that for the vertices $X_1, \ldots, X_n$ of $\tilde{\mathcal{H}}$ $(X_1, \ldots, X_n) \in V$, then these vertices can move in V *only*, i.e. neighbourhoods of the sets

   $$\{\xi \in [\tilde{\mathcal{H}}]\,;(X_1, \ldots, X_n) \in V, F(X_1, \ldots, X_n) = 0\}$$

   and

   $$\{\xi \in [\tilde{\mathcal{H}}]\,;(X_1, \ldots, X_n) \notin V, F(X_1, \ldots, X_n) = 0\},$$

   that are distinct in $[\tilde{\mathcal{H}}]$ exist.

Now we move the vertices $X_1, \ldots, X_n, F(X_1, \ldots, X_n)$ such that $(X_1, \ldots, X_n) \in V$ and fix $F(X_1, \ldots, X_n)$ in $0 \in]-L_F, L_F[$. From the above a mechanism $\mathcal{H}$ is defined, and we have with corollary 2.14

$$
\begin{aligned}
[\mathcal{H}] &= \{\xi \in [\tilde{\mathcal{H}}]\,;\, F(X_1, \ldots, X_n) = 0\} \\
&= V \amalg T_{rest}
\end{aligned}
$$

This results in a mechanical linkage $\mathcal{H}$ in the sense of definition 2.1, and V is a disjoint part of $[\mathcal{H}]$. q.e.d

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