REAL ALGEBRAIC SURFACES

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The aim of these lectures is to study real algebraic surfaces with the help of the minimal model program. I mainly concentrate on those surfaces F which are rational over $\mathbb{C}$. In some sense this is the class where the theory over $\mathbb{C}$ and over $\mathbb{R}$ differ the most.

The study of real algebraic curves and surfaces is very old, in fact it predates algebraic geometry over $\mathbb{C}$. From our point of view the first significant result was the classification of real cubic surfaces by [Schläfli1863]. [Zeuthen1874] studied plane curves of degree 4 and their bitangents, which is essentially equivalent to degree 2 Del Pezzo surfaces (though he related them to cubic surfaces only). The first systematic study was done in [Comessatti13] and [Comessatti14]. These papers presented a breakthrough whose extent has not been appreciated for over 60 years. (For instance, the main theorem of [Comessatti14] disproves a conjecture of [Nash52].) The monograph [Segre42] treats cubic surfaces in detail.

The development of the theory of surfaces over nonclosed fields was aimed primarily at arithmetic questions, but the results of [Segre51, Manin66, Iskovskikh80] are very interesting even in the real case.

A systematic study of real algebraic surfaces is given in [Silhol89]. This book develops the theory from the point of view of topology, Galois cohomology and minimal model theory.

The aim of my lectures is to investigate real algebraic surfaces using primarily the theory of minimal models. For surfaces this is mostly a question of personal preference, however, this approach becomes significant for real algebraic threefolds.

The topological methods of real algebraic geometry are basically homological. That is, they give information about the homology of the set of real points. In the surface case the set of real points is a topological manifold of real dimension 2. Compact manifolds of real dimension 2 are classsified by their homology, hence in this case there is no need for more detailed information. By contrast, for compact manifolds of real dimension 3 their homology carries relatively little information. The geometric methods of these lectures can be generalized to threefolds. Thus these notes form an introduction to my papers [Kollár97], though the 3–dimensional methods and results are never mentioned explicitly.

Notation 0.1. If X is a scheme over $\mathbb{R}$ then $X(\mathbb{R})$ denotes its set of real points with the Euclidean topology and $X_{\mathbb{C}} := X \times_{\operatorname{Spec}\mathbb{R}} \operatorname{Spec}\mathbb{C}$ denotes the complexification of X. Sometimes I write $X_{\mathbb{R}}$ instead of X to emphasize that we are over $\mathbb{R}$. One should keep in mind the distinction between $X_{\mathbb{R}}$ and $X(\mathbb{R})$. The first is a scheme over $\mathbb{R}$, thus among its closed points we find all conjugate pairs of complex points of $X_{\mathbb{C}}$, but $X(\mathbb{R})$ is the collection of the real points only.

$\mathbb{P}^n$ stands for projective n-space as a scheme over $\mathbb{Q}$. Thus $\mathbb{P}^n_{\mathbb{R}}$ is projective n-space as a scheme over $\mathbb{R}$ and $\mathbb{P}^n(\mathbb{R})$ is the set of real points, customarily denoted by $\mathbb{RP}^n$.

When talking about a point, curve or vector bundle etc. on a real variety X, these are, by definiton, objects defined over $\mathbb{R}$. I will always explicitly mention if we are considering a point, curve or vector bundle etc. on $X_{\mathbb{C}}$.

If $Z \subset X_{\mathbb{C}}$ is a subscheme, then $\bar{Z} \subset X_{\mathbb{C}}$ denotes the subscheme defined by conjugate equations. Then $Z + \bar{Z} \subset X_{\mathbb{C}}$ can be defined by real equations and it can be viewed as a subscheme of $X_{\mathbb{R}}$. That is, there is a unique subscheme $Z_{\mathbb{R}} \subset X_{\mathbb{R}}$ such that $Z_{\mathbb{R}} \times_{\operatorname{Spec}\mathbb{R}} \operatorname{Spec}\mathbb{C} = Z + \bar{Z}$.

A property of X (irreducible, normal, smooth etc.) always refers to the scheme theoretic property of X. Thus if X is smooth then it is smooth not just at all points of $X(\mathbb{R})$, but also at all complex points. When talking about irreducibility (normality etc.) of $X_{\mathbb{C}}$, I say X is "geometrically irreducible" or "irreducible over $\mathbb{C}$". In some cases this does not matter (for instance X is normal or smooth iff $X_{\mathbb{C}}$ is) but in other cases the two versions are different (X may be irreducible but not geometrically irreducible).

The logical continuation of this terminology is to say that two irreducible $\mathbb{R}$-schemes X, Y are birational iff there is a map $f : X \dashrightarrow Y$ defined over $\mathbb{R}$ which is birational. Similarly, X is rational iff it is birational to $\mathbb{P}^n_{\mathbb{R}}$ over $\mathbb{R}$. Unfortunately many authors use the word "rational" to mean "geometrically rational". To avoid confusion, I will use the expressions "rational over $\mathbb{R}$" and "rational over $\mathbb{C}$".

1. Minimal models of real algebraic surfaces

Definition 1.1. Let X be a variety over a field k. A *1–cycle* on X is a formal linear combination $C = \sum c_i C_i$, where the $C_i \subset X$ are irreducible, reduced and proper curves. A 1–cycle is called *effective* if $c_i \geq 0$ for every i.

Two 1–cycles C, C' are *numerically equivalent* if $(C \cdot D) = (C' \cdot D)$ for every Cartier divisor D on X. 1–cycles with real coefficients modulo numerical equivalence form a vectorspace, denoted by $N_1(X)$. $N_1(X)$ is finite dimensional by the Theorem of the base of Néron–Severi (cf.

[Hartshorne77, p.447]). Its dimension, denoted by $\rho(X)$, is called the *Picard number* of X.

Effective 1–cycles generate a cone $NE(X) \subset N_1(X)$. Its closure in the Euclidean topology $\overline{NE}(X) \subset N_1(X)$ is called the *cone of curves* of X.

If K_X is Cartier (or at least some multiple of K_X is Cartier) then set

$$\overline{NE}(X)_{K\geq 0} := \{z \in \overline{NE}(X) | (z \cdot K_X) \geq 0\}.$$

Let $V \subset \mathbb{R}^n$ be a closed convex cone. For $v \in V$, a ray $\mathbb{R}^{\geq 0} v \subset V$ is called *extremal* if $u, u' \in V$, $u + u' \in \mathbb{R}^{\geq 0} v$ implies that $u, u' \in \mathbb{R}^{\geq 0} v$. Intuitively: $\mathbb{R}^{\geq 0} v$ is an edge of V.

An extremal ray $\mathbb{R}^{\geq 0} z \subset \overline{NE}(X)$ is called K_X*-negative* if $(z \cdot K_X) < 0$. This does not depend on the choice of z in the ray.

Let $R \subset \overline{NE}(X)$ be a ray. A *contraction* of R is a morphism $f_R : X \to X'$ such that $(f_R)_* \mathcal{O}_X = \mathcal{O}_{X'}$ and a curve $C \subset X$ is mapped to a point iff $[C] \in R$.

Exercise 1.2. Show that the contraction of a ray is unique (if it exists). Also, if X' is projective then R is an extremal ray. Find examples of extremal rays which can not be contracted.

Exercise 1.3. Let F be a smooth projective surface and $z \in \overline{NE}(F)$ a 1–cycle such that $(z^2) > 0$. Then z or $-z$ is in the interior of $\overline{NE}(F)$. Thus if $\mathbb{R}^{\geq 0} z$ is extremal and $(z^2) > 0$ then $\overline{NE}(F)$ is 1–dimensional.

We use the following description of the cone of curves of smooth surfaces over $\mathbb{C}$. The result is essentially equivalent to the theory of minimal models of surfaces developed around the turn of the 20th century. This formulation (and its higher dimensional generalization) is due to [Mori82]. See also [Kollár96, Kollár-Mori98] for proofs.

Theorem 1.4 (Cone Theorem)**.** *Let F be a smooth projective surface over an algebraically closed field. Then there are curves $C_i \subset F$ such that*

$$\overline{NE}(F) = \overline{NE}(F)_{K\geq 0} + \sum \mathbb{R}^{\geq 0}[C_i],$$

and the $\mathbb{R}^{\geq 0}[C_i]$ are K_F-negative extremal rays of $\overline{NE}(F)$.

Moreover, we can assume that each $C_i \subset F$ is a smooth rational curve and $(C_i^2) \in \{-1, 0, 1\}$. If $(C_i^2) = 1$ (resp. $(C_i^2) = 0$) for some i then $F \cong \mathbb{P}^2$ (resp. F is a minimal ruled surface over a curve). □

Let now F be a smooth projective surface over $\mathbb{R}$. If C is a 1–cycle on $F_{\mathbb{C}}$ then $C + \bar{C}$ is a 1–cycle on F, and every 1–cycle on F arises this way, at least if we use rational or real coefficients. Thus (1.4) immediately gives:

Theorem 1.5 (Cone Theorem over $\mathbb{R}$). *Let F be a smooth projective surface over $\mathbb{R}$. Then there are smooth rational curves $C_i \subset F_{\mathbb{C}}$ with $(C_i^2) \in \{-1, 0, 1\}$ such that*

$$\overline{NE}(F) = \overline{NE}(F)_{K\geq 0} + \sum \mathbb{R}^{\geq 0}[C_i + \bar{C}_i],$$

and the $\mathbb{R}^{\geq 0}[C_i + \bar{C}_i]$ are K_F-negative extremal rays of $\overline{NE}(F)$.

Proof. There is one point that we need to be careful about. Namely, it happens frequently that C_i gives an extremal ray but $C_i + \bar{C}_i$ does not. So we have to throw away some of the C_i appearing in (1.4). □

1.6 (Geometric irreducibility). Let $X \subset \mathbb{P}^n$ be a variety over $\mathbb{C}$ and $\bar{X}$ the variety defined by conjugate equations. The disjoint union of X and $\bar{X}$ is invariant under conjugation, and so there is a real variety $Y_{\mathbb{R}}$ such that $Y_{\mathbb{C}} \cong X \cup \bar{X}$. Such real varieties are not particularly interesting since the theory of $Y_{\mathbb{R}}$ over $\mathbb{R}$ is equivalent to the theory of X over $\mathbb{C}$. Thus it is reasonable to restrict our attention to real varieties Y such that $Y_{\mathbb{C}}$ is irreducible, that is, Y is geometrically irreducible.

Of course, during a proof we may run into a subvariety of $Y_{\mathbb{R}}$ which is geometrically reducible, and these have to be dealt with appropriately. Thus we can not ignore such varieties completely.

Definition 1.7. Let S be a smooth projective surface over a field k. S is called a *Del Pezzo surface* if S is geometrically irreducible and $-K_S$ is ample. It is called *minimal* (over k) if $\rho(S) = 1$.

S, together with a morphism to a smooth curve $f : S \to B$ is called a *conic bundle* if every fiber is isomorphic to a plane conic. A conic bundle is called *minimal* if $\rho(S) = 2$.

The geometric description and meaning of the extremal rays occurring in (1.5) is given in the next result:

Theorem 1.8. *Let F be a smooth projective geometrically irreducible surface over $\mathbb{R}$ and $R \subset \overline{NE}(F)$ a K_F-negative extremal ray. Then R can be contracted $f : F \to F'$, and we obtain one of the following cases:*

(B) *(Birational) F' is a smooth projective surface over $\mathbb{R}$ and $\rho(F') = \rho(F) - 1$. F is the blow up of F' at a closed point P. We have two cases:*
   (a) *$P \in F'(\mathbb{R})$, or*
   (b) *P is a pair of conjugate points.*

(C) *(Conic bundle) $B := F'$ is a smooth curve, $\rho(F) = 2$ and $F \to B$ is a conic bundle. The fibers $f^{-1}(P) : P \in B(\mathbb{C})$ are smooth, except for an even number of ponts $P_1, \dots, P_{2m} \in B(\mathbb{R})$. $(K_F^2) = 8(1 - g(B)) - 2m$.*

(D) *(Del Pezzo surface) F' is a point, $\rho(F)=1$, $-K_F$ is ample and we have one of the following cases*
   (a) $(K_F^2)=9$ *and* $F\cong \mathbb{P}^2$.
   (b) $(K_F^2)=8$ *and* $F\cong (x_0^2+x_1^2+x_2^2-x_3^2=0)\subset \mathbb{P}^3$.
   (c) $(K_F^2)=2$.
   (d) $(K_F^2)=1$.

Proof. By (1.8), there is a curve $C\cong \mathbb{P}^1$ over $\mathbb{C}$ such that $C+\bar C$ generates R and $(C^2)\in\{-1,0,1\}$.

We consider various possibilities.

Assume first that $(C^2)=-1$. If $C=\bar C$ then the contraction of C in $F_{\mathbb{C}}$ is defined over $\mathbb{R}$, thus F is the blow up of a surface at a real point. (This is Castelnuovo's contraction theorem, cf. [Hartshorne77, V.5.7].) If C and $\bar C$ are disjoint, then we can contract them simultaneously over $\mathbb{R}$ to obtain $f:F\to F'$ which is an isomorphism near $F(\mathbb{R})$.

If $(C^2)=-1$ and $(C\cdot\bar C)=1$ or $(C^2)=(C\cdot\bar C)=0$, then $C+\bar C$ has selfintersection 0. From Riemann–Roch we obtain that

$$h^0(F,\mathcal{O}_F(m(C+\bar C))\geq \chi(F,\mathcal{O}_F(m(C+\bar C))=m+\chi(\mathcal{O}_F),$$

thus $m(C+\bar C)$ moves in a linear system for $m\gg 1$. It's moving part is base point free by (1.14). Let $f:F\to B$ be the Stein factorization of the resulting morphism.

Let $A\subset F$ be an irreducible fiber. If A is a multiple fiber, write it as $A=mA_1$. Since $[A_1]\in R$,

$$2g(A_1)-2=(A_1^2)+(A_1\cdot K_F)=(A_1\cdot K_F)<0.$$

Thus A_1 is isomorphic to a smooth conic over $\mathbb{R}$ (1.10) and $(A_1\cdot K_F)=-2$. The generic fiber A_g is not multiple, so $(mA_1\cdot K_F)=(A_g\cdot K_F)=-2$ which shows that there are no multiple fibers.

Let $A_1+A_2=f^{-1}(b)$ be a reducible fiber over $\mathbb{C}$, where A_1 is an irreducible and reduced curve. In particular, $(A_1^2)<0$. $\bar A_1+\bar A_2=f^{-1}(\bar b)$ is also a fiber. If $b\neq\bar b$ then A_1 is disjoint from $\bar A_1$, thus $((A_1+\bar A_1)^2)=(A_1^2)+(\bar A_1^2)<0$. $[A_1+\bar A_1]+[A_2+\bar A_2]\in R$ and $[A_1+\bar A_1]\notin R$, a contradiction. Thus all singular fibers lie over real points $P_1,\dots,P_r\in B(\mathbb{R})$.

Therefore, $A_1+\bar A_1\subset f^{-1}(b)$. Every fiber of f is irreducible over $\mathbb{R}$, thus $A_1+\bar A_1=f^{-1}(b)$. We get that $(A_1\cdot K_X)=-1$ and so A_1 is a -1-curve and $A_1+\bar A_1$ is isomorphic to a pair of conjugate lines in $\mathbb{P}^2$. F is a conic bundle over B by (1.11).

Over $\mathbb{C}$ we can contract one of the components of every singular fiber to obtain a minimal ruled surface. The selfintersection number of the canonical class of a minimal ruled surface is $8(1-g(B))$, and each singular fiber drops this number by 1. We see in (2.2) that the number of singular fibers is even.

Assume next that $(C^2)=-1$ and $r:=(C\cdot\bar{C})\geq 2$. Then

$$((C+\bar{C})^2)=-2+2(C\cdot\bar{C})=2r-2>0.$$

By (1.3) this implies that $\overline{NE}(F)$ is 1-dimensional, hence $-K_F\equiv a(C+\bar{C})$ for some $a>0$. $(-K_F\cdot(C+\bar{C}))=2$, thus

$$(C+\bar{C})\equiv(1-r)K_F \quad\text{and}\quad 2r-2=(1-r)^2(K_F^2).$$

This gives the possibilities $r=2,(K_F^2)=2$ or $r=3,(K_F^2)=1$.

If $(C^2)=0$ and $r:=(C\cdot\bar{C})\geq 1$ then a computation as above gives that $8=r(K_F^2)$, which allows too many cases. It is better to consider this geometrically.

By (1.4), C is a fiber of a $\mathbb{P}^1$-bundle $g:F_{\mathbb{C}}\to D$ over $\mathbb{C}$ and $\bar{C}$ is a (possibly multiple) section of g. Thus D is rational. By the classification of minimal ruled surfaces, either $F_{\mathbb{C}}\cong\mathbb{P}^1\times\mathbb{P}^1$ and we are done by (1.16), or g has a unique section E with negative selfintersection. E is then defined over $\mathbb{R}$, thus $\rho(F)=2$, a contradiction.

We are left with the case when $F_{\mathbb{C}}\cong\mathbb{CP}^2$ and C is a line in $\mathbb{CP}^2$. Then $\bar{C}$ is another line and C and $\bar{C}$ intersect in a unique point, which is therefore real. We can get another real point, and so also a real line. Thus $\mathcal{O}_F(1)$ is defined over $\mathbb{R}$ and $F\cong\mathbb{RP}^2$. $\square$

As a consequence we obtain the minimal model program (MMP for short) for real algebraic surfaces:

Theorem 1.9 (MMP for surfaces)**.** *Let F be a smooth projective geometrically irreducible surface over $\mathbb{R}$. Then there is a sequence of morphisms*

$$F=F_0\stackrel{f_0}{\to}F_1\to\cdots F_{m-1}\stackrel{f_{m-1}}{\to}F_m=F^*$$

such that each $f_i:F_i\to F_{i+1}$ is a birational contraction as in (1.8.B) and F^* satisfies precisely one of the following properties:

- (M) *(Minimal model) K_{F^*} is nef. (That is, it has nonnegative intersection number with every curve in F^*.)*
- (C) *(Conic bundle) F^* is a conic bundle over a curve $f:F^*\to B$. In particular, $\rho(F^*)=2$*
- (D) *(Del Pezzo surface) $\rho(F^*)=1$, $-K_{F^*}$ is ample and F^* is among those listed in (1.8.D).*

Proof. We do the steps of (1.8.B) as long as we can. $\rho(F_{i+1})=\rho(F_i)-1$, so eventually we reach $F^*=F_m$ where we can not perform a contraction as in (1.8.B). If K_{F^*} is nef then we have a minimal model.

If K_{F^*} is not nef, then by (1.8) we can perform a contraction as in (1.8.C–D). This gives our last two cases. □

Exercise 1.10. Let k be a field and C a smooth projective curve over k such that $C_{\bar{k}} \cong \mathbb{P}^1$. Show that C is isomorphic to a smooth conic over k. Also, $C \cong \mathbb{P}^1_k$ iff $C(k) \neq \emptyset$.

Exercise 1.11. Let S be a smooth projective surface over $\mathbb{C}$ and $f : S \to B$ a morphism to a smooth curve. Assume that $f^{-1}(b) \cong \mathbb{P}^1$ for some $b \in B$ and every fiber has at most 2 irreducible components. Show that $-K_F$ is very ample on the fibers, $f_*\mathcal{O}_F(-K_F)$ is a rank 3 vector bundle over B and we have an injection

$$F \hookrightarrow \operatorname{Proj}_B f_*\mathcal{O}_F(-K_F).$$

Under this injection the fibers of f become conics. Such a surface is called a *conic bundle* over B.

Exercise 1.12. Let X be a variety over $\mathbb{R}$ such that $X_{\mathbb{C}} \cong \mathbb{P}^n$. Show that $X \cong \mathbb{P}^n$ if n is even, but not necessarily if n is odd.

Exercise 1.13 (Cohomology commutes with base change)**.** Let X be a variety over $\mathbb{R}$ and F a coherent sheaf on X. Show that

$$H^i(X, F) \otimes_{\mathbb{R}} \mathbb{C} \cong H^i(X_{\mathbb{C}}, F_{\mathbb{C}}).$$

Exercise 1.14. Let F be a smooth projective surface and $D \subset F$ an irreducible curve such that $(D^2) = 0$. Then the moving part of $|mD|$ is either empty or base point free.

Notation 1.15. $Q^{r,s}$ denotes the quadric hypersurface $(x_1^2 + \cdots + x_r^2 - x_{r+1}^2 - \cdots - x_{r+s}^2 = 0)$. $Q^{2,1}$ is isomorphic to $\mathbb{P}^1$ by (1.10).

Lemma 1.16. *Let F be a smooth projective surface over $\mathbb{R}$ such that $F_{\mathbb{C}} \cong \mathbb{P}^1 \times \mathbb{P}^1$. Then one of the following holds:*

1. $F \cong Q^{2,2} \cong Q^{2,1} \times Q^{2,1}$, $\rho(F) = 2$ *and* $F(\mathbb{R}) \sim S^1 \times S^1$,
2. $F \cong Q^{3,1}$, $\rho(F) = 1$ *and* $F(\mathbb{R}) \sim S^2$,
3. $F \cong Q^{4,0} \cong Q^{3,0} \times Q^{3,0}$, $\rho(F) = 2$ *and* $F(\mathbb{R}) = \emptyset$,
4. $F \cong Q^{3,0} \times \mathbb{P}^1$, $\rho(F) = 2$ *and* $F(\mathbb{R}) = \emptyset$.

Proof. Let $C \subset F_{\mathbb{C}}$ be one of the rulings. Then $\bar{C}$ is another ruling, thus either $(C \cdot \bar{C}) = 0$ or $(C \cdot \bar{C}) = 1$.

If $(C \cdot \bar{C}) = 0$ then the linear system $|C + \bar{C}|$ is defined over $\mathbb{R}$ and maps F onto a conic. Similarly for the other rulings, thus F is the product of two conics. All 3 possibilities are listed.

If $(C \cdot \bar{C}) = 1$ then $\mathcal{O}_F(C + \bar{C})$ is a line bundle on F which is of type $\mathcal{O}_{F_{\mathbb{C}}}(1,1)$ over $\mathbb{C}$. Thus its global sections embed F as a quadric. $Q^{3,1}$ is the only quadric not yet accounted for. □

In the above proofs we had to establish several times that certain line bundles on $F_{\mathbb{C}}$ are defined over $\mathbb{R}$. This is frequently a quite subtle point. Some aspects of it are treated in the next exercise.

Exercise 1.17. Let X be a scheme over $\mathbb{R}$ and L a line bundle on X. Then $L_{\mathbb{C}}$ is a line bundle on $X_{\mathbb{C}}$ and $L_{\mathbb{C}} \cong \bar{L}_{\mathbb{C}}$. Thus if M is a line bundle on $X_{\mathbb{C}}$ and $M \not\cong \bar{M}$, then M is not the complexification of a real line bundle.

Find a curve C over $\mathbb{R}$ and a line bundle M on $C_{\mathbb{C}}$ such that $M \cong \bar{M}$ but M is not the complexification of a real line bundle.

Let X be a scheme over $\mathbb{R}$ and M a line bundle on $X_{\mathbb{C}}$ such that $M \cong \bar{M}$. Show that $M^{\otimes 2}$ is the complexification of a real line bundle. If X is connected, reduced and $X(\mathbb{R}) \neq \emptyset$ then M itself is the complexification of a real line bundle.

More generally, let X_K be an integral scheme defined over a field K and $L \supset K$ a Galois extension with Galois group G. Show that if X_K has a K-point then $\mathrm{Pic}(X_K) = \mathrm{Pic}(X_L)^G$.

2. The topology of $F(\mathbb{R})$

In this section we study the MMP from the topological point of view. The main results of this section are already in [Comessatti14].

Notation 2.1. $M \uplus N$ denotes the disjoint union of M and N. $\uplus rN$ denotes the disjoint union of r copies of N. $M\#N$ denotes the connected sum of two manifolds M and N (which are assumed to have the same dimension). $\#rN$ denotes the connected sum of r copies of N. (By definition, $\#0M = S^{\dim M}$.) $M \sim N$ denotes that M and N are homeomorphic.

One can give a complete topological description of the various contractions in (1.8):

Theorem 2.2. *Let F be a smooth projective geometrically irreducible surface over $\mathbb{R}$ and $R \subset \overline{NE}(F)$ a K_F-negative extremal ray. The following is the topological description of the corresponding contraction:*

- (B) *(Birational) F is the blow up of F' at a closed point P. We have two cases:*
  - (a) *If $P \in F'(\mathbb{R})$ then $F(\mathbb{R}) \sim F'(\mathbb{R})\#\mathbb{RP}^2$.*
  - (b) *If P is a pair of conjugate points then $F(\mathbb{R}) \sim F'(\mathbb{R})$.*
- (C) *(Conic bundle) $f : F \to B$ is a conic bundle with singular fibers $f^{-1}(P_1), \dots, f^{-1}(P_{2m})$. Then*

$$F(\mathbb{R}) \sim \uplus mS^2 \uplus N_1 \uplus \cdots \uplus N_b,$$

where b is the number of connected components of $B(\mathbb{R})$ which do not contain any of the points P_i and each N_i is either a torus or a Klein bottle.

(D) *(Del Pezzo surface) There are 4 cases:*
(a) *If $(K_F^2) = 9$ then $F(\mathbb{R}) \sim \mathbb{RP}^2$.*
(b) *If $(K_F^2) = 8$ then $F(\mathbb{R}) \sim S^2$.*
(c) *If $(K_F^2) = 2$ then $F(\mathbb{R}) \sim \uplus 4S^2$.*
(d) *If $(K_F^2) = 1$ then $F(\mathbb{R}) \sim \mathbb{RP}^2 \uplus 4S^2$.*

Proof. Blowing up replaces a point with all tangent directions through that point. So we remove a disc and put in an interval bundle over S^1 whose boundary is connected. This is a Möbius strip and so $F(\mathbb{R}) \sim F'(\mathbb{R})\#\mathbb{RP}^2$.

In the conic bundle case, let $M \sim S^1$ be a connected component of $B(\mathbb{R})$. If none of the P_i lie on M then $F(\mathbb{R}) \to B(\mathbb{R})$ is a smooth S^1-bundle over M, this gives either a torus or a Klein bottle. If k of the points $P_1, \dots, P_k \in M \sim S^1$ correspond to singular fibers then, after reindexing, they divide M into k intervals $[P_i, P_{i+1}]$ (subscript mod k). $F(\mathbb{R})$ is alternatingly empty or a copy of S^2 over the intervals. Thus k is even.

In the Del Pezzo case we are done if $(K_F^2) = 9, 8$. The cases $(K_F^2) = 2, 1$ are considerably harder. They follow from (6.3) and (6.8). □

Using (2.2) it is easy to determine which 2-manifolds occur as $F(\mathbb{R})$ for geometrically rational surfaces F. The conclusion is that orientable surfaces of genus > 1 do not occur. This is the main result of [Comessatti14].

Theorem 2.3. *Let F be a smooth, projective surface over $\mathbb{R}$ such that $F_{\mathbb{C}}$ is rational. Then one of the following holds:*

0. *$F(\mathbb{R}) = \emptyset$.*
1. *$F(\mathbb{R}) \sim S^1 \times S^1$.*
2. *$F(\mathbb{R}) \sim \#r_1\mathbb{RP}^2 \uplus \cdots \uplus \#r_m\mathbb{RP}^2$ for some $r_1, \dots, r_m \geq 0$.*

All these cases do occur.

Proof. Apply the MMP over $\mathbb{R}$ to get $F = F_1 \to F_2 \to \cdots$. We prove the theorem by induction on the number of blow ups in the sequence. If $F_i \to F_{i+1}$ is the inverse of the blowing up of a real point, then $F_i(\mathbb{R}) \sim F_{i+1}(\mathbb{R})\#\mathbb{RP}^2$. If $F_i \to F_{i+1}$ is the inverse of the blowing up of a pair of conjugate points, then $F_i(\mathbb{R}) \sim F_{i+1}(\mathbb{R})$. The induction works since $(S^1 \times S^1)\#\mathbb{RP}^2 \sim \#3\mathbb{RP}^2$.

Thus we are reduced to one of the following two cases:

1. F has a conic bundle structure $F \to B$, or
2. F is Del Pezzo and $\rho(F) = 1$.

In the first case, $B_{\mathbb{C}} \cong \mathbb{CP}^1$ since $F_{\mathbb{C}}$ is rational. Thus either $B(\mathbb{R}) = \emptyset$ and so $F(\mathbb{R}) = \emptyset$, or $B \cong \mathbb{RP}^1$. Thus $F(\mathbb{R})$ is the torus or the Klein bottle if there are no singular fibers and $F(\mathbb{R}) \sim \uplus m S^2$ if there are $2m > 0$ singular fibers by (2.2.C). Note that $S^2 = \#0\mathbb{RP}^2$ by convention.

In the second case we use (2.2.D). □

Exercise 2.4. Let X and Y be smooth projective varieties over $\mathbb{R}$. Assume that X and Y are birational to each other (over $\mathbb{R}$). Show that $X(\mathbb{R})$ and $Y(\mathbb{R})$ have the same number of connected components.

2.5 (Vector bundles over real varieties)**.** Let X be a veriety over $\mathbb{R}$ and $p : V \to X$ a vector bundle of rank n. Locally V is like $U \times \mathbb{A}^n_{\mathbb{R}} \to U$ where $U \subset X$ is Zariski open and $\mathbb{A}^n_{\mathbb{R}} = \operatorname{Spec}_{\mathbb{R}} \mathbb{R}[t_1, \dots, t_n]$ is affine n-space over $\mathbb{R}$. (Which should *not* be identified with $\mathbb{R}^n$!)

As usual, to V one can associate a vector bundle $p_{\mathbb{C}} : V_{\mathbb{C}} \to X_{\mathbb{C}}$ and also a real vector bundle $p(\mathbb{R}) : V(\mathbb{R}) \to X(\mathbb{R})$ which is obtained by taking the $\mathbb{R}$-valued points of $\mathbb{A}^n_{\mathbb{R}}$ which is exactly $\mathbb{R}^n$.

(To complete the picture, any real vector bundle on a manifold can be complexified, and $V(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} \cong V_{\mathbb{C}}|_{X(\mathbb{R})}$.)

2.6 (Degrees of line bundles over $\mathbb{R}$ and $\mathbb{C}$)**.**

Let B be a smooth projective curve over $\mathbb{C}$ and L a line bundle on B. Let s be a nonzero meromorphic section of L. The number of zeros minus the number of poles of s on B (counted with multiplicity) is called the *degree of* L. Let Y be a smooth projective variety over $\mathbb{C}$ and L a line bundle on Y. For any curve $B \subset Y$ the degree of $L|_B$ is defined. It is also called the *intersection number* of B and L and denoted by $(B \cdot L)$.

Let $A \sim S^1$ be a compact 1–dimensional manifold and L a real line bundle on M. Let s be a nonzero section of L. The number of zeros of s on A (counted with multiplicity) makes sense only mod 2. If M is a compact manifold and L a real line bundle on M then for any 1-cycle $A \subset M$ we obtain the $\mathbb{Z}_2$-valued *intersection number* of A and L. It is denoted by $(A \cap L)$. (To be precise, I should write $(A \cap w_1(L))$ where $w_1(L)$ stands for the first Stiefel–Whitney class of L. This is a class in $H^1(X(\mathbb{R}), \mathbb{Z}_2)$ analogous to the first Chern class of a complex line bundle, cf. [Milnor-Stasheff74, Sec. 4].)

Let now X be a smooth projective variety over $\mathbb{R}$, $C \subset X$ a curve and L a line bundle on X. We obtain two numbers:

$$(L(\mathbb{R}) \cap C(\mathbb{R})) \quad \text{and} \quad (C_{\mathbb{C}} \cdot L_{\mathbb{C}}).$$

What is the relationship between them?

To answer this, take a real meromorphic section s of L which has only finitely many zeros and poles on C. When we count the real zeros and poles of s on $C(\mathbb{R})$, we miss the complex zeros and poles of s on $C_{\mathbb{C}}$. Since s is real, the complex zeros and poles come in conjugate pairs. Thus we conclude that

$$(C(\mathbb{R}) \cap L(\mathbb{R})) \equiv (C_{\mathbb{C}} \cdot L_{\mathbb{C}}) \mod 2,$$

which is best possible since the left hand side is defined only mod 2 anyhow.

2.7 (Orientability of $X(\mathbb{R})$ and the canonical class)**.**
Let M be a differentiable manifold, $0 \in M$ a point and $x_1, \dots, x_n$ local coordinates. A *local orientation* of M at 0 is a choice of an n-form $f(x)dx_1 \wedge \cdots \wedge dx_n$ with $f(0) \neq 0$ up to multiplication by a positive function. An *orientation* of M is a nowhere zero global n-form on M, up to multiplication by a positive function. n-forms are sections of the real line bundle $\det T^*_M$. If $S^1 \sim A \subset M$ is a loop then one can choose a consistent oreintation of M along $A \Leftrightarrow \det T^*_M$ has a nowhere zero section along $A \Leftrightarrow (\det T^*_M \cap A) = 0$.
If X is a smooth variety over $\mathbb{R}$ then n-forms appear as sections of the canonical line bundle. This proves that

$$\det T^*_{X(\mathbb{R})} \cong K_X(\mathbb{R}).$$

In many cases this gives a way to decide if $X(\mathbb{R})$ is orientable or not.

Corollary 2.8. *Let X be a smooth projective variety over $\mathbb{R}$. Assume that there is a curve $C \subset X$ such that $(C \cdot K_X)$ is odd. Then $X(\mathbb{R})$ is not orientable.*

Proof. We have proved above that

$$(\det T^*_{X(\mathbb{R})} \cap C(\mathbb{R})) \equiv (C \cdot K_X) \equiv 1 \mod 2.$$

$C(\mathbb{R})$ may have several components, but along one of them $\det T^*_{X(\mathbb{R})}$ has odd degree, so we can not choose a consistent orientation along that component. □

Exercise 2.9. Show that $\mathbb{RP}^n$ is orientable iff n is odd. Let $X \subset \mathbb{P}^n$ be a smooth hypersurface of degree d. Show that $X(\mathbb{R})$ is orientable if $n-d$ is odd. Show that $X(\mathbb{R})$ is not orientable if n and d are both odd. If n and d are both even, then $X(\mathbb{R})$ may or may not be orientable.

If $(C \cdot K_X)$ is even, then it can happen that $X(\mathbb{R})$ is not orientable along an even number of components of $C(\mathbb{R})$. In some cases we are still able to conclude orientability of $X(\mathbb{R})$ using stronger assumptions:

Exercise 2.10. Let X be a smooth projective variety over $\mathbb{R}$. Assume that $K_X \cong L^{\otimes 2}$ for a real line bundle L. Show that $X(\mathbb{R})$ is orientable.

More generally, assume that $K_X \cong \mathcal{O}_X(2D + D')$ where D, D' are divisors over $\mathbb{R}$ and $D'(\mathbb{R})$ has codimension at least 2 in $X(\mathbb{R})$. (This is equivalent to assuming that every irreducible component of D' is geometrically reducible.) Show that $X(\mathbb{R})$ is orientable.

3. Birational classification

Definition 3.1. Let F be a smooth real algebraic surface. A surface obtained from F by blowing up a real points and b pairs of conjugate complex points (possibly infinitely near) is denoted by $(F, a, 2b)$.

Given F and a, b, the surfaces of the form $(F, a, 2b)$ consitute a connected family if $F(\mathbb{R})$ and $F_{\mathbb{C}}$ are both connected.

Lemma 3.2. *We have the following elementary birational equivalences between the minimal models in (1.8).*

1. $(\mathbb{P}^2, 2, 0) \cong (Q^{2,2}, 1, 0)$.
2. $(\mathbb{P}^2, 0, 2) \cong (Q^{3,1}, 1, 0)$.
3. $(Q^{4,0}, 0, 2)$ *is isomorphic to the blow up of* $Q^{3,0} \times \mathbb{P}^1$ *at a pair of conjugate points on the same section* $Q^{3,0} \times P$, $P \in \mathbb{RP}^1$.
4. *Any minimal conic bundle over a rational curve with* 2 *singular fibers is isomorphic to* $(Q^{3,1}, 0, 2)$.

Proof. In the first two cases we blow up the 2 points in $\mathbb{P}^2$ and then contract the line through them to get a quadric.

$Q^{4,0} \cong Q^{3,0} \times Q^{3,0}$, let π_1 be the first projection. The pencil of planes through the 2 points gives a map $p : Q^{4,0} \dashrightarrow \mathbb{P}^1$.

$$(\pi_1, p) : Q^{4,0} \dashrightarrow Q^{3,0} \times \mathbb{P}^1 \quad \text{is birational}$$

and becomes a morphism after blowing up the 2 points.

Finally assume that $F \to B$ is a minimal conic bundle over a rational curve with 2 singular fibers. By (1.8.C), $B(\mathbb{R}) \neq \emptyset$, thus $B \cong \mathbb{P}^1$. $F_{\mathbb{C}}$ is the blow up of a minimal ruled surface F'' at 2 points. We can even assume that F'' has a section E with negative selfintersection $(E^2) = -k$ and the two points are not on E. If $k \geq 2$ then all other sections of F'' have selfintersection at least 2, so $E \subset F$ is the unique section with negative selfintersection. Thus E is defined over $\mathbb{R}$ and $F \to B$ is not minimal.

Thus $k = 1$ and there is a unique section $E' \subset F''$ such that $({E'}^2) = 1$ and E' passes through the two blown up points. Let $\bar{E} \subset F$ be the birational transform. Then E and $\bar{E}$ have to be conjugate. Contracting them gives the quadric $Q^{3,1}$. $\square$

Lemma 3.3. *Let F be a smooth projective surface over $\mathbb{R}$ such that $F_{\mathbb{C}}$ is a minimal ruled surface over $\mathbb{CP}^1$. Then one of the following holds:*

1. *$F_{\mathbb{C}} \cong \mathbb{P}^1 \times \mathbb{P}^1$ (these cases were enumerated in (1.16)), or*
2. *$F \cong \operatorname{Proj}_B(\mathcal{O}_B + \mathcal{O}_B(-r))$ is a minimal ruled surface over a smooth real conic B for some $r > 0$.*

Proof. By the classification of minimal ruled surfaces, either $F_{\mathbb{C}} \cong \mathbb{P}^1 \times \mathbb{P}^1$, or $F_{\mathbb{C}}$ has a unique irreducible curve E with negative selfintersection $-r$. E and the ruling $g : F \to B$ are then defined over $\mathbb{R}$ and $g_*\mathcal{O}_F(E) \cong \mathcal{O}_B + \mathcal{O}_B(-r)$. □

As a corollary, we obtain the following birational classification of real surfaces such that $F_{\mathbb{C}}$ is rational:

Corollary 3.4. *Let F be a smooth real projective surface such that $F_{\mathbb{C}}$ is rational. Then F is birationally equivalent over $\mathbb{R}$ to a surface in exactly one of the following classes:*

1. *$Q^{3,0} \times \mathbb{P}^1$. In this case $F(\mathbb{R}) = \emptyset$.*
2. *$\mathbb{P}^2$. In this case $F(\mathbb{R})$ is connected.*

3_m. *Minimal conic bundle with $2m$ ($m \geq 2$) singular fibers. In this case $F(\mathbb{R})$ has m connected components.*

4. *Degree 2 minimal Del Pezzo surface.*
5. *Degree 1 minimal Del Pezzo surface.*

Remark. In (6.3) and (6.8) we prove that $F(\mathbb{R})$ has 4 (resp. 5) connected components if F is a minimal Del Pezzo surface of degree 2 (resp. 1).

Proof. Let $F \to F^*$ be the minimal model of F. By (1.8) F^* is either one of those listed above, or F^* is a conic bundle with 0 or 2 singular fibers. The former are treated in (3.3). The latter are birational to $Q^{3,0} \times \mathbb{P}^1$ by (3.2).

The number of connected components of the real part is a birational invariant (2.4), hence the cases (1–3_m) are all different birationally.

The cases (4–5) differ birationally from the other ones by (3.5). □

We use, without proof, the following result about the birational classification of low degree Del Pezzo surfaces over any field. Lectures 2–3 of [Kollár-Smith97] serve as a good introduction.

Theorem 3.5. [Segre51, Manin66] *Let k be a field (of characteristic zero) and F a minimal Del Pezzo surface of degree 1,2 or 3 over k. Then*

1. *F is not rational (over k),*
2. *F is not birational (over k) to any conic fibration,*

3. *F is birational to another minimal Del Pezzo surface F' of degree 1,2 or 3 over k iff F is isomorphic to F'.* □

This theorem, (3.4) and (2.4) imply the following:

Corollary 3.6. [Silhol89, VI.6.5] *Let F be a smooth projective surface over $\mathbb{R}$. The following are equivalent:*

1. *F is birational to $\mathbb{P}^2$ over $\mathbb{R}$.*
2. *$F_{\mathbb{C}}$ is birational to $\mathbb{CP}^2$ and $F(\mathbb{R})$ is connected.* □

4. Birational Classification of Conic Fibrations

Definition 4.1. Let F be a smooth projective surface over a field k. A morphism $f : F \to B$ to a smooth curve is called a *conic fibration* if the generic fiber is isomorphic to a plane conic (over $k(B)$). By (1.10) this is equivalent to assuming that $f^{-1}(b) \cong \mathbb{P}^1_{\bar{k}}$ for a general $b \in B(\bar{k})$.

In this section we discuss the birational classification of those surfaces over $\mathbb{R}$ which admit a conic fibration. This covers all surfaces where the MMP ends with the case (1.9.C).

This is done in two steps. First we consider those birational maps which preserve the conic fibration. To be precise:

Definition 4.2. Two conic fibrations $f : F \to B$ and $f' : F' \to B'$ are called *birational* if there is a birational map $\phi : F \dashrightarrow F'$ and an isomorphism $\tau : B \cong B'$ (both over k) which give a commutative diagram

$$\begin{array}{ccc} F & \overset{\phi}{\dashrightarrow} & F' \\ f\downarrow & & \downarrow f' \\ B & \overset{\tau}{\dashrightarrow} & B' \end{array}$$

The second step is to understand the birational maps between F and F' which do not preserve the conic fibration. Fortunately, in many cases there are no such maps. (For a proof see [Silhol89, V–VI] or the original paper of [Iskovskikh67].)

Theorem 4.3. *Let k be a field and $f : F \to B$ a relatively minimal conic bundle over B. Let $f' : F' \to B'$ be any conic fibration and $\phi : F \dashrightarrow F'$ any birational map (over k). Then*

1. *If $(K_F^2) \le 0$ then ϕ is a birational map of the conic fibrations.*
2. *If $(K_F^2) = 2$ then F and F' are birational conic fibrations (though ϕ itself need not respect the fibration structure).* □

Definition 4.4. Let $f : F \to B$ be a conic fibration over $\mathbb{R}$. The image of the set of real points $f(F(\mathbb{R})) \subset B(\mathbb{R})$ is a union of finitely many closed intervals. Let us denote it by $I(F)$.

The main theorem of the section shows that $I(f)$ characterizes f:

Theorem 4.5. *Two conic fibrations $f : F \to \mathbb{P}^1$ and $f' : F' \to \mathbb{P}^1$ over $\mathbb{R}$ are birational iff there is an isomorphism $\tau : \mathbb{P}^1 \cong \mathbb{P}^1$ such that $\tau(I(f)) = I(f')$.*

Proof. Let $\phi : F \dashrightarrow F'$ and $\tau : \mathbb{P}^1 \cong \mathbb{P}^1$ be a birational map of the two conic fibrations. Then $F'(\mathbb{R})$ and $\phi(F(\mathbb{R}))$ agree outside finitely many fibers, thus $I(f')$ and $\tau(I(f))$ differ only at finitely many points. Unions of closed intervals can not differ at finitely many points only, thus in fact $\tau(I(f)) = I(f')$.

The converse is established by bringing each conic fibration to a normal form. (The roots a_i in (4.6) are the boundary points of $I(f)$. This leaves two choices for $I(f)$ itself, corresponding to the two choices of the sign on the right hand side.) $\square$

Theorem 4.6. *Let $f : F \to \mathbb{P}^1$ be a conic fibration over $\mathbb{R}$. Then f is birational to a conic fibration $f' : F' \to \mathbb{P}^1$ with affine equation*

$$x^2 + y^2 = \pm \prod_{i=1}^{2m} (z - a_i) \subset \mathbb{A}^3,$$

where the a_i are distinct real numbers.

The proof rests on the following simple lemma about quadratic forms:

Lemma 4.7. *Let k be a field (of characteristic different from 2) and $Q(x_0, \dots, x_n)$ a quadratic form over k which is anisotropic (that is $Q = 0$ has no nontrivial solution over k). For any $a \in k$ the following are equivalent*

1. *$Q = 0$ has a nontrivial solution over $k(\sqrt{a})$.*
2. *After a suitable coordinate change, Q can be written as $b(y_0^2 - a y_1^2) + Q'(y_2, \dots, y_n)$.*

Proof. (2) $\Rightarrow$ (1) is shown by the substitution $\mathbf{y} := (\sqrt{a} : 1 : 0 \cdots : 0)$.

Conversely, assume that $\mathbf{v} \in k(\sqrt{a})^{n+1}$ satisfies $Q(\mathbf{v}) = 0$. Let $\bar{\mathbf{v}}$ denote the conjugate of $\mathbf{v}$. Then $Q(\bar{\mathbf{v}}) = 0$. $\mathbf{v}$ and $\bar{\mathbf{v}}$ span a 2-dimensional linear subspace of $k(\sqrt{a})^{n+1}$ which is defined over k. That is, there is a linear subspace $V \subset k^{n+1}$ such that $V \otimes_k k(\sqrt{a}) = \langle \mathbf{v}, \bar{\mathbf{v}} \rangle$. Q is nondegenerate on V (since $Q = 0$ has no solutions in k), thus $k^{n+1} = V + V^{\perp}$ where $V^{\perp}$ is the orthogonal complement of V with respect to Q. Let $y_2, \dots, y_n$ be coordinates on $V^{\perp}$ and choose coordinates y_0, y_1 on V such that

$$\mathbf{v} = (\sqrt{a}, 1) \quad \text{and} \quad \bar{\mathbf{v}} = (-\sqrt{a}, 1). \quad \square$$

4.8 (Proof of (4.6)).

Let $k := \mathbb{R}(t)$ be the quotient field of $\mathbb{P}^1_{\mathbb{R}}$. The generic fiber of $F \to \mathbb{P}^1$ is birational to a plane conic C_k over k (1.10).

If C_k has a k-point (equivalently, if $F \to \mathbb{P}^1$ has a section) then F is birational to $\mathbb{P}^1 \times \mathbb{P}^1$ by (1.10). C_k has a point over $k(\sqrt{-1}) = \mathbb{C}(t)$ (equivalently, $F \to \mathbb{P}^1$ has a section over $\mathbb{C}$). Thus by (4.7), in suitable coordinates the equation of C_k becomes $x_0^2 + x_1^2 = g(t)x_2^2$ for some $g(t) \in \mathbb{R}(t)$.

We can multiply through with the square of the denominator of g, thus we may assume that $g(t) \in \mathbb{R}[t]$.

Write $g = f(t)g_1(t)^2 \prod_a (z-a)(z-\bar{a})$ where f has only simple real roots and the a are nonreal complex numbers. We can divide by $g_1(t)^2$. If $a = u + iv$ then $(z-a)(z-\bar{a}) = (z-u)^2 + v^2$. Note that

$$\begin{aligned} g(t)(h_0^2 + h_1^2)x_2^2 &= x_0^2 + x_1^2 \quad \text{is equivalent to} \\ g(t)x_2^2 &= \left(\frac{x_0h_0 - x_1h_1}{h_0^2 + h_1^2}\right)^2 + \left(\frac{x_0h_1 + x_1h_0}{h_0^2 + h_1^2}\right)^2. \end{aligned}$$

Using this, we can get rid of the complex factors $\prod_a (z-a)(z-\bar{a})$ one at a time. At the end we obtain the required normal form, except that we may have an odd number of factors on the right hand side:

$$x^2 + y^2 = \pm \prod_{i=1}^{2m-1} (z - a_i).$$

In this case we first apply a translation to ensure that 0 is not among the a_i and then make a substitution $(x, y, z) \mapsto (xz^{-n}, yz^{-n}, z^{-1})$ to get the equation

$$x^2 + y^2 = \pm \prod_{i=1}^{2m} (z - a_i'),$$

where $a_i' = a_i^{-1}$ for $i < 2m$ and $a_{2m}' = 0$. □

Putting things together, we obtain the following criterion for birational equivalence of conic fibrations. The result corrects a slight inaccuracy in [Silhol89, VI.3.15].

Exercise 4.9. Two conic bundles

$$\begin{aligned} F &= (x^2 + y^2 = c\prod_{i=1}^{2m}(z - a_i)) \to \mathbb{P}^1 \quad \text{and} \\ F' &= (x^2 + y^2 = c'\prod_{i=1}^{2m}(z - a_i')) \to \mathbb{P}^1 \end{aligned}$$

are birational to each other iff there is a permutation $\sigma \in S_{2m}$ and a matrix

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in GL(2,\mathbb{R}) \quad \text{such that} \quad a'_{\sigma(i)} = \frac{\alpha a_i + \beta}{\gamma a_i + \delta}, \quad \text{and}$$

c' and $c\prod_i(\gamma a_i + \delta)$ have the same sign.

Exercise 4.10. Using elementary transformations of conic bundles, give a geometric proof of the results in this section.

4.11 (Moduli of conic fibrations)**.**
Let F be a smooth projective surface over $\mathbb{R}$ which admits a conic fibration $f : F \to \mathbb{P}^1$. We proved that if $I(f) \subset \mathbb{RP}^1$ has at least 3 components then $I(f)$ (modulo the action of $GL(2,\mathbb{R})$) determines F up to birational equivalence (over $\mathbb{R}$).

The space of m disjoint closed intervals in $\mathbb{RP}^1$ is a connected manifold of real dimension $2m$. The quotient by the $GL(2,\mathbb{R})$ action gives a $2m-3$ dimensional topological space (it has some quotient singularities). With some more care, we could even realize this space as the set of real points of a $(2m-3)$-dimensional algebraic variety.

For $m = 0$ there are 2 conic fibrations up to birational equivalence: $\mathbb{P}^1 \times \mathbb{P}^1$ gives $I(f) = \mathbb{RP}^1$ and $Q^{3,0} \times \mathbb{P}^1$ gives $I(f) = \emptyset$. For $m = 1$ we have only one birational equivalence class by (3.2.4).

For $m = 2$ we see in (5.8) that all such surfaces are birational to each other (though they are not birational as conic fibrations).

5. Del Pezzo Surfaces of Degree ≥ 3

In this section we describe all Del Pezzo surfaces of degree $d \geq 3$ over $\mathbb{R}$.

Exercise 5.1. Let $g : S \to S'$ be a birational morphism of smooth surfaces. Show that if H is ample on S then $f(H)$ is ample on S'. Thus if S is Del Pezzo then S' is also Del Pezzo.

Proposition 5.2. *Let F be a smooth real Del Pezzo surface which is birational to $Q^{3,0} \times \mathbb{P}^1$. Set $d := (K_F^2)$. Then $d \in \{8, 6, 4, 2\}$.*

If $d = 8$ then F is isomorphic to either $Q^{3,0} \times Q^{3,0}$ or to $Q^{3,0} \times \mathbb{P}^1$.

If $d < 8$ then F is isomorphic to to $Q^{3,0} \times \mathbb{P}^1$ blown up in $\frac{1}{2}(8-d)$ pairs of conjugate points.

Therefore, for $d \in \{6, 4, 2\}$ such surfaces form a connected family.

Proof. Apply the MMP over $\mathbb{R}$ to obtain $F \to \cdots \to F^*$. If F is Del Pezzo then so is F^* by (5.1), and $(K_{F^*}^2) \geq (K_F^2)$. Hence in our case F^* is either $Q^{3,0} \times Q^{3,0}$ or $Q^{3,0} \times \mathbb{P}^1$. By (3.2.3) any blow up of $Q^{3,0} \times Q^{3,0}$ at a pair of conjugate points is also a blow up of $Q^{3,0} \times \mathbb{P}^1$. $\square$

Proposition 5.3. *Let F be a smooth real Del Pezzo surface which is birational to $\mathbb{P}^2$. Set $d := (K_F^2)$. Then $9 \geq d \geq 1$ and we have one of the following cases:*

If $d = 9$ then F is isomorphic to $\mathbb{P}^2$.

If $d = 8$ then F is isomorphic to either $Q^{3,1}$ or to $Q^{2,2}$ or to $\mathbb{P}^2$ blown up at a real point.

If $d < 8$ then F is isomorphic to one of the following:

1. *$\mathbb{P}^2$ blown up at $a \geq 0$ real points and $b \geq 0$ pairs of conjugate points for some $a + 2b = 9 - d$. Thus $F(\mathbb{R}) \sim \#(a+1)\mathbb{RP}^2$.*
2. *$Q^{3,1}$ blown up at $b = \frac{1}{2}(8-d)$ pairs of conjugate points (so d is even). Thus $F(\mathbb{R}) \sim S^2$.*
3. *$Q^{2,2}$ blown up at $b = \frac{1}{2}(8-d)$ pairs of conjugate points (so d is even). Thus $F(\mathbb{R}) \sim S^1 \times S^1$.*

   Therefore, for any $d < 8$, such surfaces with a given topological type $F(\mathbb{R})$ form a connected family.

Proof. The minimal model of such a surface is either $\mathbb{P}^2$, $Q^{3,1}$ or $Q^{2,2}$. By (3.2.1–2) any blow up of $Q^{3,1}$ or to $Q^{2,2}$ at a real point is also a blow up of $\mathbb{P}^2$. □

The two propositions above account for all Del Pezzo surfaces of degrees $d \geq 5$. The results are summarized in the next statement:

Corollary 5.4. *The following table lists all topological types of the real points of Del Pezzo surfaces of degrees $9 \geq d \geq 5$. All surfaces of a fixed degree and topological type form a connected family, except for $d = 8$ and $F(\mathbb{R}) = \emptyset$ when there are 2 such surfaces.*

degree	*topological types*
9	$\mathbb{RP}^2$
8	S^2 *or* $S^1 \times S^1$ *or* $\mathbb{RP}^2\#\mathbb{RP}^2$ *or* $\emptyset$
7	$\mathbb{RP}^2$ *or* $\#3\mathbb{RP}^2$
6	S^2 *or* $S^1 \times S^1$ *or* $\mathbb{RP}^2\#\mathbb{RP}^2$ *or* $\#4\mathbb{RP}^2$ *or* $\emptyset$
5	$\mathbb{RP}^2$ *or* $\#3\mathbb{RP}^2$ *or* $\#5\mathbb{RP}^2$

The following result shows that odd degree Del Pezzo surfaces over $\mathbb{R}$ are relatively easy to understand:

Lemma 5.5. *Every degree $2d-1$ Del Pezzo surface F over $\mathbb{R}$ with $\rho(F) \geq 2$ is the blow up of a degree $2d$ Del Pezzo surface at a real point.*

Proof. Since (K_F^2) is odd, F is not a minimal conic bundle. Thus F is either the blow up of a degree $2d$ Del Pezzo surface at a real point or the blow up of a degree $2d+1$ Del Pezzo F' surface at a conjugate pair of complex points $P + \bar{P}$.

If $F' \cong \mathbb{P}^2$ then let $L \subset \mathbb{P}^2$ be the line through the two points. Its birational transform on F is a line.

Otherwise, F' is again not minimal by (1.8), hence F' contains a line L over $\mathbb{R}$ by induction. $P, \bar{P} \notin L$ since otherwise the birational transform of L on F would have a nonnegative intersection number with K_F. Thus L gives a real line on F.

Contracting a real line on F we get a degree $2d$ Del Pezzo surface. □

This shows that the study of degree 3 Del Pezzo surfaces is reduced to the study of degree 4 cases. The classification of these two classes is summarized next. These results were obtained by [Schläfli1863], who actually worked directly with cubic surfaces.

Corollary 5.6. *The following table lists all topological types of the real points of Del Pezzo surfaces of degrees 4 and 3. All surfaces of a fixed degree and topological type form a connected family.*

degree	*topological types*
4	S^2 *or* $S^1 \times S^1$ *or* $\mathbb{RP}^2 \# \mathbb{RP}^2$ *or* $\#4\mathbb{RP}^2$ *or* $\emptyset$ *or* $S^2 \uplus S^2$
3	$\mathbb{RP}^2$ *or* $\#3\mathbb{RP}^2$ *or* $\#5\mathbb{RP}^2$ *or* $\#7\mathbb{RP}^2$ *or* $S^2 \uplus \mathbb{RP}^2$

Moreover, in the $S^2 \uplus S^2$ *case the monodromy interchanges the two components.*

Proof. As we noted above, it is sufficient to describe all degree 4 Del Pezzo surfaces.

If a degree 4 Del Pezzo surface F is obtained from a higher degree surface by blowing up then we are reduced to (5.4). Otherwise F is a conic bundle over $\mathbb{P}^1$ with 4 singular fibers. These 4 singular fibers give 8 lines on F. By looking at the set of all lines over $\mathbb{C}$, we see that the remaining 8 lines again form 4 pairs and determine another morphism to $\mathbb{P}^1$. Thus F is a double cover of $\mathbb{P}^1 \times \mathbb{P}^1$ ramified along a curve $D \subset \mathbb{P}^1 \times \mathbb{P}^1$ of type $(2, 2)$. D has 4 horizontal and 4 vertical tangents and the 16 lines are sitting over these tangents.

The rest is a special case of (5.7). □

Exercise 5.7. Show that the space of all smooth real curves of type $(2, 2)$ on $\mathbb{P}^1 \times \mathbb{P}^1$ has 7 connected components. They are determined by the homotopy classes of the components of $D(\mathbb{R})$: $\emptyset$ or $(0, 0)$ or $(0, 0) \uplus (0, 0)$ or $(1, 1) \uplus (1, 1)$ or $(1, -1) \uplus (1, -1)$ or $(1, 0) \uplus (1, 0)$ or $(0, 1) \uplus (0, 1)$.

Exercise 5.8. [Silhol89, VI.3.5] Using the correspondence between $(2, 2)$-curves on $\mathbb{P}^1 \times \mathbb{P}^1$ and degree 4 Del Pezzo surfaces show that any two minimal conic bundles F, F' with 4 singular fibers are birational, by producing examples $S \to \mathbb{P}^1 \times \mathbb{P}^1$ such that one conic bundle structure of S is birational to F and the other to F'. (This should be easier after the next section.)

Exercise 5.9. [Schläfli1863] Show that a smooth cubic over $\mathbb{R}$ has 27,15,7 or 3 real lines.

6. Del Pezzo Surfaces of Degree 2 and 1

Notation 6.1. Let $D \subset \mathbb{P}^2$ be a degree 4 smooth real curve. $D(\mathbb{R})$ divides $\mathbb{RP}^2$ into connected open sets and precisely one of these is nonorientable (denoted by U_D). We choose an equation $f(x,y,z) \in \mathbb{R}[x,y,z]$ of D such that f is negative on U_D.

We can associate two different degree 2 Del Pezzo surfaces to D. One is $F_D^+ := (u^2 = f(x,y,z) \subset \mathbb{P}^3(1,1,1,2)$ and the other $F_D^- := (u^2 = -f(x,y,z) \subset \mathbb{P}^3(1,1,1,2)$.

The correspondence $F_D^+ \leftrightarrow F_D^-$ is a natural involution on the space of degree 2 real Del Pezzo surfaces.

D has 28 bitangents over $\mathbb{C}$ and over each bitangent of D we get a pair of lines on $F_D^{\pm}$. This gives a total of 56 lines.

The topological classification of degree 4 plane curves over $\mathbb{R}$ is very old, it is already contained in [Plücker1839]. (See [Viro90] for a recent survey of the study of low degree real plane curves.) This implies the topological classification of degree 2 real Del Pezzo surfaces. The following proposition summarizes these results.

Proposition 6.2. *There are 6 topological types of degree 4 smooth real plane curves. Correspondingly there are 12 topological types of degree 2 real Del Pezzo surfaces. The following table gives the complete list. The types in the same row correspond to each other under* $D \leftrightarrow F_D^+ \leftrightarrow F_D^-$.

$D(\mathbb{R})$	$F_D^+(\mathbb{R})$	$F_D^-(\mathbb{R})$
○ ○ ○ ○	$\uplus 4S^2$	$\#8\mathbb{RP}^2$
○ ○ ○	$\uplus 3S^2$	$\#6\mathbb{RP}^2$
○ ○	$S^2 \uplus S^2$	$\#4\mathbb{RP}^2$
○	S^2	$\#2\mathbb{RP}^2$
$\emptyset$	$\emptyset$	$\mathbb{RP}^2 \uplus \mathbb{RP}^2$
◎	$S^1 \times S^1$	$S^2 \uplus \#2\mathbb{RP}^2$

[Zeuthen1874] studied the bitangents of degree 4 plane curves. He proved the equivalence of (6.3.1) and (6.3.5). He understood the relationship between degree 4 plane curves and cubic surfaces. (Projecting a cubic surface from one of its points, the branch curve is a plane quartic. Equivalently, blowing up the cubic at a point we get a degree 2 Del Pezzo surface.) This is, however, not the natural thing to do from the modern viewpont. Most of (6.3) is proved in [Comessatti13].

Theorem 6.3. *Let $D \subset \mathbb{P}^2$ be a degree 4 smooth real curve. The following are equivalent:*

1. *All 28 bitangents of D are real.*
2. *All 56 lines of F_D^- are real.*
3. *F_D^- is isomorphic to $\mathbb{P}^2$ blown up in 7 real points.*
4. *$F_D^-(\mathbb{R}) \sim \#8\mathbb{RP}^2$.*
5. *$D(\mathbb{R}) \sim \uplus 4S^1$.*
6. *$F_D^+(\mathbb{R}) \sim \uplus 4S^2$.*
7. *F_D^+ has Picard number 1 over $\mathbb{R}$.*

Proof. (1) $\Rightarrow$ (2): A neighborhood of a line in $\mathbb{RP}^2$ is not orientable, thus any bitangent is contained in U_D (except for the points of tangency). f is negative on any bitangent and so $u^2 = -f$ has real solutions, giving 56 real lines on F_D^-.

(2) $\Rightarrow$ (3): Over $\mathbb{C}$, F_D^- is the blow up of $\mathbb{P}^2$ at 7 points, hence it has 7 disjoint lines. If all lines are real, we have 7 disjont real lines. Contracting these we get a Del Pezzo surface of degree 9 over $\mathbb{R}$. By (1.8.D) it is $\mathbb{P}^2_{\mathbb{R}}$.

(3) $\Rightarrow$ (4): Topologically, each blowing up is connected sum with $\mathbb{RP}^2$.

(4) $\Rightarrow$ (5): This follows from (6.2).

(5) $\Rightarrow$ (6): This also follows from (6.2).

(6) $\Rightarrow$ (7): Assume to the contrary that F_D^+ has Picard number ≥ 2 over $\mathbb{R}$. By (1.8) we have one of 2 cases:

1. F_D^+ is a minimal conic bundle with 6 singular fibers. In this case $F_D^+(\mathbb{R}) \sim \uplus 3S^2$, a contradiction.
2. F_D^+ is the blow up of a Del Pezzo surface of degree 3 or 4 over $\mathbb{R}$. By (5.6) $F_D^+(\mathbb{R})$ has at most 2 connected components, a contradiction.

(7) $\Rightarrow$ (1): Assume that D has a complex bitangent L. Its conjugate $\bar{L}$ is again a bitangent. Let $C \subset F_D^+$ be a complex line over L. Its conjugate $\bar{C}$ lies over $\bar{L}$. Then $(C\cdot\bar{C}) \leq 1$ (the only possible intersection point lies over $L \cap \bar{L}$). Thus F_D^+ has either a disjoint pair of conjugate lines or a conic bundle structure, a contradiction. ☐

Proposition 6.4. [Klein1876] *The space of degree 4 smooth real plane curves has 6 connected components corresponding to the 6 topological types in (6.2). The space of degree 2 real Del Pezzo surfaces has 12 connected components corresponding to the 12 topological types in (6.2).*

Proof. The two parts are equivalent and it is sufficient to treat the F_D^- cases. D always has a real bitangent (6.5), thus F_D^- contains a real line. So F_D^- is obtained by blowing up a degree 3 Del Pezzo surface at a real point. (6.4) now follows from (5.6). ☐

Exercise 6.5. [Zeuthen1874] Let C be a smooth real plane curve of degree 4. Let d be the number of ovals of $D(\mathbb{R})$ which are not contained in another oval. Show that C has $4+2d(d-1)$ real bitangents.

6.6 (Degree 1 Del Pezzo surfaces)**.**

Let F be a degree 1 Del Pezzo surface over any field k. $|-K_F|$ is a pencil with exactly one base point. So this is a k-point and $F(k)\neq\emptyset$. $|-2K_F|$ is base point free and exhibits F as a double cover of a quadric cone $Q\subset\mathbb{P}^3$, ramified along a curve $D\subset Q$ which is a complete intersection of Q with a cubic surface with equation $(f=0)$. D does not pass through the vertex of the cone.

$F_{\bar{k}}$ contains 240 lines (that is -1-curves); cf. [Manin72, IV.4.3]. We obtain these as follows. Take a plane $H\subset\mathbb{P}^3$ which is tangent to D at 3 points. The preimage of $H\cap Q$ in F has 2 irreducible components, each is a line. Thus we conclude that there are 120 planes which are tangent to D at 3 points.

Assume now that $k=\mathbb{R}$. Since $Q(\mathbb{R})\neq\emptyset$, we can write $Q=(x^2+y^2=1)$ in suitable affine coodinates (x,y,z) on $\mathbb{A}^3$. That is, $Q(\mathbb{R})$ is a cylinder with a singular point at infinity.

As in the degree 2 case, for each (nonhomogeneous) cubic $f(x,y,z)$ we obtain two degree 1 Del Pezzo surfaces, given by affine equations

$$F_f^{\pm}:=(x^2+y^2-1=u^2\mp f(x,y,z)=0)\subset\mathbb{A}^4.$$

There are 2 types of simple closed loops on a cylinder: null homotopic ones (I call them ovals) and those homotopic to a plane section (I call them big circles).

Since $D(\mathbb{R})$ is the intersection of a cylinder with a cubic, it has 3 or 1 intersection points with any ruling line of the cylinder. Thus $D(\mathbb{R})$ contains either 3 big circles (and no ovals) or 1 big circle. D has genus 4, hence by Harnack's theorem (cf. [Shafarevich72, VII.4]), $D(\mathbb{R})$ has at most 5 connected components. An oval can not be inside another oval since this would give 4 points on a ruling. Furthermore, we can not have an oval on one side the big circle and at least two ovals on the other side. Indeed, choosing points P_1,P_2,P_3 inside the 3 ovals, there is a plane H through them. Then H intersects each oval in at least 2 points, and also the big circle. So $(H\cdot D)\geq 8$, but D has degree 6, a contradiction.

If all the ovals are on the same side of the big circle, we can normalize f so that it is positive on the other side. The other cases are symmetrical and it makes little sense to normalize f.

We can summarize these results in the following table:

Proposition 6.7. *There are 7 topological types of degree 6 smooth real complete intersection curves on the cylinder $(x^2+y^2=1)$, not passing through the point at infinity. Correspondingly there are 11 topological types of degree 1 real Del Pezzo surfaces. The following table gives the complete list. The types in the same row correspond to each other under $D=(f=0)\cap Q \leftrightarrow F_f^+ \leftrightarrow F_f^-$.*

$D(\mathbb{R})$	$F_f^+(\mathbb{R})$	$F_f^-(\mathbb{R})$
1 big circle + 4 ovals	$\mathbb{RP}^2 \uplus 4S^2$	$\#9\mathbb{RP}^2$
1 big circle + 3 ovals	$\mathbb{RP}^2 \uplus 3S^2$	$\#7\mathbb{RP}^2$
1 big circle + 2 ovals	$\mathbb{RP}^2 \uplus 2S^2$	$\#5\mathbb{RP}^2$
1 big circle + 1 oval	$\mathbb{RP}^2 \uplus S^2$	$\#3\mathbb{RP}^2$
1 big circle + 0 oval	$\mathbb{RP}^2$	$\mathbb{RP}^2$
1 big circle + 1+1 ovals	$\#3\mathbb{RP}^2 \uplus S^2$	$\#3\mathbb{RP}^2 \uplus S^2$
3 big circles	$\mathbb{RP}^2 \uplus \#2\mathbb{RP}^2$	$\mathbb{RP}^2 \uplus \#2\mathbb{RP}^2$

The following theorem, due to [Comessatti13], is the degree 1 version of (6.3). I thank F. Russo for checking the arguments of Comessatti.

Theorem 6.8. *Let $D=(f=0)\subset Q$ be a degree 6 smooth real complete intersection curve on the cylinder $Q=(x^2+y^2=1)$. The following are equivalent:*

1. *All 120 triple tangents of D are real and f is negative on all of them.*
2. *All 240 lines of F_f^- are real.*
3. *F_f^- is isomorphic to $\mathbb{P}^2_{\mathbb{R}}$ blown up in 8 real points.*
4. *$F_f^-(\mathbb{R}) \sim \#9\mathbb{RP}^2$.*
5. *$D(\mathbb{R}) \sim \uplus 5S^1$.*
6. *$F_f^+(\mathbb{R}) \sim \mathbb{RP}^2 \uplus 4S^2$.*
7. *F_f^+ has Picard number 1 over $\mathbb{R}$.*

Proof. (1) $\Rightarrow$ (2): If f is negative on a triple tangent then $u^2=-f$ has real solutions, giving a pair of real lines on F_f^-.

(2) $\Rightarrow$ (3): Over $\mathbb{C}$, F_f^- is the blow up of $\mathbb{P}^2$ at 8 points. Thus it has 8 disjoint lines. If all lines are real, we have 8 disjont real lines. Contracting these we get a Del Pezzo surface of degree 9 over $\mathbb{R}$. By (1.8.D) it is $\mathbb{P}^2_{\mathbb{R}}$.

(3) $\Rightarrow$ (4): Topologically, each blowing up is connected sum with $\mathbb{RP}^2$.

(4) $\Rightarrow$ (5): This follows from (6.7).

(5) $\Rightarrow$ (6): This also follows from (6.7).

(6) $\Rightarrow$ (7): Assume to the contrary that F_f^+ has Picard number ≥ 2 over $\mathbb{R}$. F_f^+ can not be a minimal conic bundle since (K^2) is odd. Thus

F_f^+ is the blow up of Del Pezzo surface of degree 2 or 3 over $\mathbb{R}$. By (6.2, 5.6) $F_f^+(\mathbb{R})$ has at most 4 connected components, a contradiction.

(7) $\Rightarrow$ (1): If D has a complex triple tangent, we can argue as in (6.3.(7) $\Rightarrow$ (1)) that F_f^+ contains a conjugate pair of lines $C, \bar{C}$ such that $(C \cdot \bar{C}) \leq 2$. $(C + \bar{C}) \equiv rK$ for some $r \in \mathbb{Z}$, thus $2(C \cdot \bar{C}) - 2 = r^2$. This is impossible.

If there is a real triple tangent such that f is positive on it then as in (1) $\Rightarrow$ (2) we get real lines on F_f^+. □

Proposition 6.9. *The space of degree 1 real Del Pezzo surfaces has 11 connected components corresponding to the 11 topological types in (6.7).*

Proof. For the nonminimal ones, this follows from (6.9) and (5.5). The minimal ones are in one–to-one correspondence with the blow ups of $\mathbb{P}^2$ at 8 real points, this is again a connected space. □

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7. Hints to selected exercises

(1.2) Blow up ≥ 10 general points (over $\mathbb{C}$) on a smooth plane cubic. The birational transform of the cubic generates an extremal ray which is not contractible. To see this show that Pic of the blown up surface injects into Pic of the cubic.

(1.3) Write down Riemann–Roch for mz and $-mz$ to get the first part. Then use this for $z + \epsilon z'$ for any $z' \in N_1(F)$. (cf. [Hartshorne77, V.1.8].)

(1.10) $|-K|$ embeds C as a conic.

(1.12) Let $H \in X_{\mathbb{C}}$ be a hyperplane. Show that $H \cap \bar{H}$ is real and use induction. Even degree symmetric powers of the empty conic give examples.

(1.13) This is a special case of [Hartshorne77, III.9.3].

(1.17) The empty conic gives a good example.

For the rest, the key point is to understand that we know more than the existence of an isomorphism $\tau : L_{\mathbb{C}} \cong \bar{L}_{\mathbb{C}}$. Namely, by conjugation this induces $\bar{\tau} : \bar{L}_{\mathbb{C}} \cong \bar{\bar{L}}_{\mathbb{C}} \cong L_{\mathbb{C}}$, and the composite of these two gives the identity of $L_{\mathbb{C}}$ (and not just an isomorphism of $L_{\mathbb{C}}$ to itself). Thus we have to choose a specific isomorphism $\tau : M \cong \bar{M}$. If X has a real point P, then on the fiber over P we can choose τ to be conjugation (and not just some constant times conjugation).

Once things are set up right, the real sections of M are those sections s of M such that $\tau(s) = \bar{s}$.

(2.4) Use the fact that a birational map between projective varieties is defined outside a codimension 2 subset.

(2.10) Let $S^1 \sim A \subset X(\mathbb{R})$ be any loop. Perturb it to achieve that A intersects $D(\mathbb{R})$ transversally at smooth points and is disjoit from $D'(\mathbb{R})$.

(4.9) z is transformed by the inverse of the matrix. Then do the explicit computation.

(4.10) Let $F \to \mathbb{P}^1$ be a minimal conic bundle. There are 2 types of elementary transformations: blow up a real point in a fiber and then contract the fiber, or blow up conjugate points in conjugate fibers and then contract the fibers.

Pick any section C over $\mathbb{C}$. Using elementary transformations get to the situation when C and $\bar{C}$ are disjoint. Show that $(C^2) = (\bar{C}^2) = -m$ if there are $2m$ singular fibers. The normal form is an affine piece of representing F as a conic bundle in $\operatorname{Proj} f_* \mathcal{O}_F(C + \bar{C})$.

(5.7) The case when $D(\mathbb{R}) = \emptyset$ is easy. There are many ways to study the remaining cases. For instance, pick a point $P \in D(\mathbb{R})$ and blow it up. By contracting the birational transforms of the two sections through P, we obtain a correspondance between pairs $(P \in D(\mathbb{R}))$ and triplets $(Q_1, Q_2 \in E(\mathbb{R}))$ where $E \subset \mathbb{P}^2$ is an elliptic curve. One has to be a little careful since $\mathbb{RP}^2$ is not orientable. If we fix the orientations of the two copies of $\mathbb{RP}^1$ in $\mathbb{RP}^1 \times \mathbb{RP}^1$ then they give local orientations of $\mathbb{RP}^2$ at the points Q_1, Q_2. There is an ambiguity of changing both orientations (since this does not change the orientation of $\mathbb{RP}^1 \times \mathbb{RP}^1$).

We have to study various cases according to the location of Q_1, Q_2 on $E(\mathbb{R})$. Moreover, we have to see how the local orientations match up if we move from Q_1 to Q_2 along $E(\mathbb{R})$. If Q_1, Q_2 are both on a pseudo line, then the two local orientations are consistent if we move in one direction and inconsistent in the other direction. However, if Q_1, Q_2 are both on an oval, then the two local orientations are either consistent in both directions or inconsistent if both directions. This gives 2 cases.

(5.8) Given a ramification curve $D \subset \mathbb{P}^1 \times \mathbb{P}^1$ we get 2 different degree 4 Del Pezzo surfaces. We want one surface S^+ where none of the lines are real. Then, in the other surface S^-, all lines are real. One can construct S^- as follows. Start with $\mathbb{P}^1 \times \mathbb{P}^1$, points $P_1, \dots, P_4$ in the first factor and $P'_1, \dots, P'_4$ in the second factor. Blow up the 4 points (P_i, P'_i). Show that we get a degree 4 Del Pezzo surface iff there is no isomorphism $\tau : \mathbb{P}^1 \times \mathbb{P}^1$ such that $\tau(P_i) = P'_i$.

Projecting to the first factor gives a conic fibration with singular fibers over P_i. Projection to the second factor is not the right thing to do. Instead, the second conic fibration is given by the linear system of curves of type $(2,2)$ passing through the 4 points (P_i, P'_i). Also keep in mind that we have to take care not only of the 4 points but also the set $I(f)$.

(5.9) and (6.5) can both be seen from the classification. One has to prove that we can not blow up a point on a line. [Schläfli1863] proved first that a cubic can be written as $C_1 - C_2 = 0$ where C_i are products of linear factors and then studied the various cases when the linear factors are all real or some are conjugates. [Zeuthen1874] notes that 2 ovals have 4 tangents, thus we need to show that there are 4 more which are either tangents to the same oval or at complex points. He does this by a continuity argument. This is a bit tricky since these 4 tangents are not invariant under deformations of the curve, just their number is.

References

BCR87. J. Bochnak, M. Coste and M-F. Roy, Géométrie algébrique réelle, Springer 1987

Comessatti13. A. Comessatti, Fondamenti per la geometria sopra superfizie razionali dal punto di vista reale, Math. Ann. 73 (1913) 1-72

Comessatti14. A. Comessatti, Sulla connessione delle superfizie razionali reali, Annali di Math. 23(3) (1914) 215-283

Hartshorne77. R. Hartshorne, Algebraic Geometry, Springer, 1977

Iskovskikh67. V. A. Iskovskikh, Rational surfaces with a pencil of rational curves, Math. USSR Sb. 3 (1967) 563-587

Iskovskikh80. V. A. Iskovskikh, Minimal models of rational surfaces over arbitrary fields, Math. USSR Izv. 14 (1980) 17-39

Klein1876. F. Klein, Über eine neuer Art von Riemannschen Flächen, Math. Ann. 10 (1876) 398–416, Reprinted in : F. Klein, Gesammelte Mathematische Abhandlungen, Springer, 1922, vol. II.

Kollár96. J. Kollár, Rational Curves on Algebraic Varieties, Springer Verlag, Ergebnisse der Math. vol. 32, 1996

Kollár97. J. Kollár, Real Algebraic Threefolds I–IV. (in preparation)

Kollár-Mori98. J. Kollár - S. Mori, Birational geometry of algebraic varieties, Cambridge Univ. Press, 1998 (to appear)

Kollár-Smith97. J. Kollár and K. Smith, Rational and Non-rational Algebraic Varieties (e-prints: alg-geom/9707013)

Manin66. Yu. I. Manin, Rational surfaces over perfect fields (in Russian), Publ. Math. IHES 30 (1966) 55-114

Manin72. Yu. I. Manin, Cubic forms (in Russian), Nauka, 1972. English translation: North-Holland, 1974, second expanded edition, 1986

Milnor-Stasheff74. J. Milnor and J. Stasheff, Characteristic Classes, Princeton Univ. Press, 1974

Mori82. S. Mori, Threefolds whose Canonical Bundles are not Numerically Effective, Ann. of Math. 116 (1982) 133-176

Nash52. J. Nash, Real algebraic manifolds, Ann. Math. 56 (1952) 405-421

Plücker1839. J. Plücker, Theorie der algebraischen Kurven, Bonn, 1839

Schläfli1863. L. Schläfli, On the distribution of surfaces of the third order into species, Phil. Trans. Roy. Soc. London, 153(1863)193-241. Reprinted in : L. Schläfli, Gesammelte Mathematische Abhandlungen, Birkhäuser, 1953, vol. II.

Segre42. B. Segre, The non-singular cubic surfaces, Clarendon Press, 1942

Segre51. B. Segre, The rational solutions of homogeneous cubic equations in four variables, Notae Univ. Rosario 2 (1951) 1-68

Shafarevich72. R. I. Shafarevich, Basic Algebraic Geometry (in Russian), Nauka, 1972. English translation: Springer, 1977, second expanded edition, 1994

Silhol84. R. Silhol, Real algebraic surfaces with rational or elliptic fibering, Math. Zeitschr. 186(1984) 465-499

Silhol89. R. Silhol, Real algebraic surfaces, Springer Lecture Notes vol. 1392, 1989

Viro90. O. Ya. Viro, Real algebraic plane curves, Leningrad Math. J. 1 (1990) 1059-1134

Zeuthen1874. H.G. Zeuthen, Sur les différentes formes des courbes du quatrième ordre, Math. Ann. 7(1874) 410-432

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