CLASSIFICATION OF IRREDUCIBLE HARISH-CHANDRA MODULES FOR $\mathrm{SL}(2,\mathbb{C})$

Dragan Miličić

Department of Mathematics, University of Utah

In this note we discuss the relationship between the geometric classification of irreducible Harish-Chandra modules using localization theory of Bernstein and Beilinson, the Langlands classification and Vogan-Zuckerman classification for the group $\mathrm{SL}(2,\mathbb{C})$.

This is an illustration of the general methods of [2]. This note should be read as a companion to [3] where all unexplained notation is introduced.

1. Cohomology groups of standard Harish-Chandra sheaves for $\mathrm{SL}(2,\mathbb{C})$. Let $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C}) \times \mathfrak{sl}(2,\mathbb{C})$ and $X = \mathbb{P}^1 \times \mathbb{P}^1$ its flag variety. Let $G = \mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})$ and $\sigma(x,y) = (y,x)$, $x, y \in G$, an involution on G.

The Cartan algebra $\mathfrak{h}$ is two dimensional and can be identified with the product of two copies of Cartan algebras of $\mathfrak{sl}(2,\mathbb{C})$. Let (α,β) be the corresponding pair of simple roots in the root system Σ in $\mathfrak{h}^*$.

Let K be the fixed point set of σ. Then K is the diagonal subgroup of G and it is isomorphic to $\mathrm{SL}(2,\mathbb{C})$. The pair $(\mathfrak{g}, K)$ is a Harish-Chandra pair. The group K acts on X with two orbits — the closed orbit C which is the diagonal in X, and its complement, the open orbit O.

The open orbit O is a Langlands orbit in X and the closed orbit C is a Zuckerman orbit.

1.1. Lemma. *The orbit O is an affine variety.*

Proof. Let $(x,y) \in O$. Then the stabilizer of (x,y) in $K = \mathrm{SL}(2,\mathbb{C})$ is the intersection $B_x \cap B_y$ of two Borel subgroups B_x and B_y in $\mathrm{SL}(2,\mathbb{C})$. Since $x \neq y$, this stabilizer is a torus T in $\mathrm{SL}(2,\mathbb{C})$. Hence, the quotient $K/T \cong O$ is affine. $\square$

As we remarked in the above proof, the stabilizer in K of a point in O is a torus in $\mathrm{SL}(2,\mathbb{C})$. Moreover, the stabilizer in K of a point in the closed orbit C is a Borel subgroup of $\mathrm{SL}(2,\mathbb{C})$. Hence, all stabilizers are connected, and for any $\lambda \in \mathfrak{h}^*$ there exist at most one irreducible K-homogeneous connection on the orbit compatible with $\lambda + \rho$. Hence, we denote the corresponding standard Harish-Chandra sheaves by $\mathcal{I}(O,\lambda)$ and $\mathcal{I}(C,\lambda)$.

Since O is affine, all higher cohomology modules of the standard sheaves $\mathcal{I}(O,\lambda)$ vanish for arbitrary $\lambda \in \mathfrak{h}^*$.

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The roots in Σ are complex. Since O is the Langlands orbit, $\sigma_O(\alpha) = -\beta$. Let $\lambda \in \mathfrak{h}^*$. Then

$$\lambda = \tfrac{1}{2}(\alpha^\vee(\lambda)\alpha + \beta^\vee(\lambda)\beta),$$

hence

$$\lambda + \sigma_O\lambda = \frac{\alpha^\vee(\lambda) - \beta^\vee(\lambda)}{2}\alpha + \frac{\beta^\vee(\lambda) - \alpha^\vee(\lambda)}{2}\beta,$$

and a K-homogeneous connection compatible with $\lambda + \rho$ exists on O if and only if $\alpha^\vee(\lambda) - \beta^\vee(\lambda) \in \mathbb{Z}$. This is equivalent to $\alpha^\vee(\lambda + \sigma_O\lambda) \in \mathbb{Z}$. Therefore, the following diagram shows the possible $\operatorname{Re}\lambda$ for which the K-homogeneous connections on O exist.

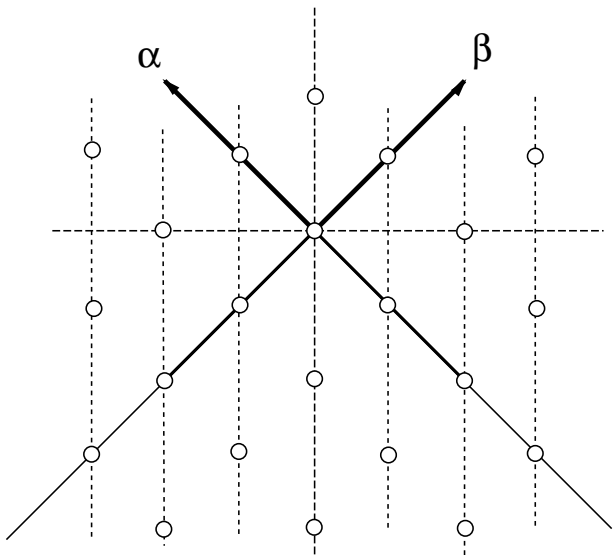

Figure 1. Standard Harish-Chandra sheaves on the open orbit

The Cartan involution σ_O corresponds to the reflection along the horizontal axis. The vertical axis is the set of real points of $\mathfrak{a}_O$ and the horizontal axis is the set of real points of $\mathfrak{t}_O$. The vertical dotted lines represents possible $\operatorname{Re}\lambda$. The standard sheaf $\mathcal{I}(O, \lambda)$ is reducible if and only if $\alpha^\vee(\lambda) \in \mathbb{Z}$ and $\beta^\vee(\lambda) \in \mathbb{Z}$, i.e., if λ is a weight. Hence, the reducibility points correspond to the white dots in the diagram. If λ is a weight, the invertible $\mathcal{O}_X$-module $\mathcal{O}(\lambda + \rho)$ is the unique irreducible submodule of $\mathcal{I}(O, \lambda)$.

Analogous discussion applies to the closed orbit C. The standard sheaf $\mathcal{I}(C, \lambda)$ exists if and only if

$$\alpha^\vee(\lambda + \sigma_C\lambda) \in \mathbb{Z}.$$

Therefore, we have the following diagram.

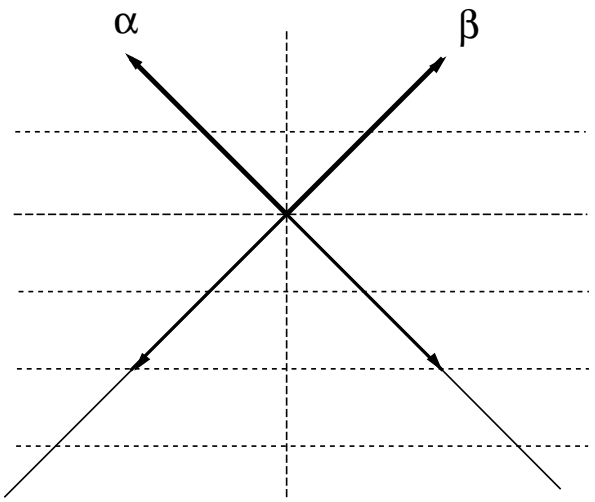

Figure 2. Standard Harish-Chandra sheaves on the closed orbit

The Cartan involution σ_C corresponds to the reflection along the vertical axis. The vertical axis is the set of real points of $\mathfrak{t}_C$ and the horizontal axis is the set of real points of $\mathfrak{a}_C$. The horizontal dotted lines represents possible $\operatorname{Re}\lambda$. Since C is closed, $\mathcal{I}(C,\lambda)$ is always irreducible.

For both orbits, if λ is integral with respect to the root α (or β), λ is a weight. This implies that either the Weyl group orbit of λ consists of antidominant elements, or λ is a weight.

If λ is not a weight, the intertwining functors I_{s_α} and I_{s_β} are equivalences of categories. Moreover,

$$I_{s_\alpha}(\mathcal{I}(C,\lambda)) = \mathcal{I}(O, s_\alpha\lambda),$$

$$I_{s_\alpha}(\mathcal{I}(O,\lambda)) = \mathcal{I}(C, s_\alpha\lambda),$$

and

$$I_{s_\beta}(\mathcal{I}(C,\lambda)) = \mathcal{I}(O, s_\beta\lambda),$$

$$I_{s_\beta}(\mathcal{I}(O,\lambda)) = \mathcal{I}(C, s_\beta\lambda).$$

This implies that

$$\Gamma(X,\mathcal{I}(C,\lambda)) = \Gamma(X,\mathcal{I}(O,s_\alpha\lambda)) = \Gamma(X,\mathcal{I}(O,s_\beta\lambda))$$

and

$$\Gamma(X,\mathcal{I}(O,\lambda)) = \Gamma(X,\mathcal{I}(C,s_\alpha\lambda)) = \Gamma(X,\mathcal{I}(C,s_\beta\lambda)).$$

This leads to

$$\Gamma(X,\mathcal{I}(O,\lambda)) = \Gamma(X,\mathcal{I}(O,-\lambda))$$

and

$$\Gamma(X,\mathcal{I}(C,\lambda)) = \Gamma(X,\mathcal{I}(C,-\lambda)).$$

If λ is a weight, the short exact sequence

$$0 \to \mathcal{O}(\lambda+\rho) \to \mathcal{I}(O,\lambda) \to \mathcal{I}(C,\lambda) \to 0$$

together with 1 leads to a long exact sequence of cohomology groups

$$\begin{aligned}(*)\quad 0 \to \Gamma(X,\mathcal{O}(\lambda+\rho)) &\to \Gamma(X,\mathcal{I}(O,\lambda)) \to \Gamma(X,\mathcal{I}(C,\lambda)) \\ &\to H^1(X,\mathcal{O}(\lambda+\rho)) \to 0 \to H^1(X,\mathcal{I}(C,\lambda)) \\ &\to H^2(X,\mathcal{O}(\lambda+\rho)) \to 0 \to H^2(X,\mathcal{I}(C,\lambda)) \to 0.\end{aligned}$$

Therefore,

$$H^2(X,\mathcal{I}(C,\lambda)) = 0$$

and

$$H^1(X,\mathcal{I}(C,\lambda)) \cong H^2(X,\mathcal{O}(\lambda+\rho)).$$

Using the Borel-Weil-Bott theorem, this implies that all higher cohomologies of $\mathcal{I}(C,\lambda)$ vanish if λ is not a regular dominant weight.

Therefore, we have proved the following vanishing theorem for cohomology of standard sheaves.

1.2. Proposition.

(i) $H^p(X,\mathcal{I}(O,\lambda)) = 0$ *for* $p > 0$.

(ii) $H^p(X,\mathcal{I}(C,\lambda)) = 0$ *for* $p > 0$ *and* λ *which is not a regular dominant weight. If* λ *is a regular dominant weight,* $H^p(X,\mathcal{I}(C,\lambda)) = 0$ *for* $p > 1$ *and* $H^1(X,\mathcal{I}(C,\lambda))$ *is the irreducible finite-dimensional module with highest weight* $\lambda - \rho$.

Now we want to discuss the global sections of standard sheaves for λ a weight. By $(*)$ we have the exact sequence

$$0 \to \Gamma(X,\mathcal{O}(\lambda+\rho)) \to \Gamma(X,\mathcal{I}(O,\lambda)) \to \Gamma(X,\mathcal{I}(C,\lambda)) \to H^1(X,\mathcal{O}(\lambda+\rho)) \to 0.$$

If λ is a singular weight, by the Borel-Weil-Bott theorem, we have

$$\Gamma(X,\mathcal{I}(O,\lambda)) = \Gamma(X,\mathcal{I}(C,\lambda)).$$

If λ is a regular weight, we denote by θ the Weyl group orbit of λ and by μ the unique antidominant element of θ. There are exactly two irreducible objects in $\mathcal{M}(\mathcal{D}_\lambda, K)$: $\mathcal{L}(O,\lambda) = \mathcal{O}(\lambda+\rho)$ and $\mathcal{L}(C,\lambda) = \mathcal{I}(C,\lambda)$. Hence, by the equivalence of categories for $\lambda = \mu$, there are exactly two irreducible objects in $\mathcal{M}(\mathcal{U}_\theta, K)$: one of them is the irreducible finite-dimensional G-module $\Gamma(X,\mathcal{O}(\mu+\rho))$ with the lowest weight $\mu+\rho$ which we denote by $F_{\mu+\rho}$. The other one is infinite dimensional module $\Gamma(X,\mathcal{I}(C,\mu))$ and we denote it by V_μ. Since $-\theta = \theta$, both modules are isomorphic to their contragredients. This implies that, by dualizing in the category of Harish-Chandra modules, the following assertions are equivalent.

1.3. Lemma.

(i) $\dim \mathrm{Ext}^1_{\mathcal{M}(\mathcal{U}_\theta,K)}(F_{\mu+\rho}, V_\mu) = 1.$

(ii) $\dim \mathrm{Ext}^1_{\mathcal{M}(\mathcal{U}_\theta,K)}(V_\mu, F_{\mu+\rho}) = 1.$

Proof. Let

$$0 \to F_{\mu+\rho} \to V \to V_\mu \to 0$$

be a short exact sequence in $\mathcal{M}(\mathcal{U}_\theta, K)$. By localizing, we get the short exact sequence

$$0 \to \mathcal{O}(\mu+\rho) \to \Delta_\mu(V) \to \mathcal{I}(C,\mu) \to 0$$

in $\mathcal{M}(\mathcal{D}_\mu, K)$. Therefore, $\Delta_\mu(V)|O = \mathcal{O}(\mu+\rho)|O$ is an irreducible K-homogeneous connection on O compatible with $\mu+\rho$. By adjointness of the direct image and the restriction it follows that there exists a nontrivial morphism i of $\Delta_\mu(V)$ into $\mathcal{I}(O,\mu)$. If V is a nontrivial extension, $\Delta_\mu(V)$ is a nontrivial extension. Hence, either $\mathcal{O}(\mu+\rho)$ is the kernel of i or i is a monomorphism. In the first case, the image of i is isomorphic with $\mathcal{I}(C,\mu)$. But this is impossible, since $\mathcal{O}(\mu+\rho)$ is the unique irreducible submodule of $\mathcal{I}(O,\mu)$. Hence, i is a monomorphism. Since $\Delta_\mu(V)$ and $\mathcal{I}(O,\mu)$ have the same length (equal to 2), i is an isomorphism. It follows that $\Gamma(X, \mathcal{I}(O,\mu))$ is the unique non trivial extension, up to an isomorphism. □

Let I_μ be the Harish-Chandra module in $\mathcal{M}(\mathcal{U}_\theta, K)$ such that

$$0 \to F_{\mu+\rho} \to I_\mu \to V_\mu \to 0$$

is exact and it doesn't split. Then, by 3, I_μ is unique up to an isomorphism. Let $\tilde{I}_\mu$ be its contragredient. Then

$$0 \to V_\mu \to \tilde{I}_\mu \to F_{\mu+\rho} \to 0$$

is exact and it doesn't split. Again, $\tilde{I}_\mu$ is unique up to an isomorphism.

Assume that λ is antidominant. The global sections of $\mathcal{I}(C,\lambda)$ are equal to V_λ, and

$$\Gamma(X, \mathcal{I}(C,\lambda)) \cong \Gamma(X, \mathcal{I}(O,\lambda))/F_{\lambda+\rho}.$$

Hence $\Gamma(X, \mathcal{I}(O,\lambda)) \cong I_\lambda$.

If λ is regular and neither dominant nor antidominant, there exists a simple root γ such that $s_\gamma\lambda$ is antidominant, i.e., $\mu = s_\gamma\lambda$. In this situation, we have

$$\mathcal{I}(O,\lambda) = I_{s_\gamma}(\mathcal{I}(C, s_\gamma\lambda))$$

and

$$L^{-1} I_{s_\gamma}(\mathcal{I}(C, s_\gamma\lambda)) = 0.$$

This implies that

$$\Gamma(X, \mathcal{I}(O,\lambda)) = \Gamma(X, \mathcal{I}(C, s_\gamma\lambda)) = V_\mu.$$

Moreover, by the above exact sequence and the Borel-Weil-Bott theorem we see that $\Gamma(X, \mathcal{I}(C,\lambda))$ is a reducible $\mathcal{U}_\theta$-module with irreducible quotient $F_{\mu+\rho}$ and irreducible

submodule V_μ. Since the $\Gamma(X, \mathcal{I}(C, \lambda))$ must be indecomposable, we see that $\Gamma(X, \mathcal{I}(C, \lambda))$ is isomorphic to $\tilde{I}_\mu$.

If λ is a regular dominant weight, from the above exact sequence we see that

$$\Gamma(X, \mathcal{I}(O, \lambda)) = \Gamma(X, \mathcal{I}(C, \lambda)).$$

Also, $s_\alpha \lambda$ is α-antidominant, hence as above we have

$$\mathcal{I}(O, \lambda) = I_{s_\alpha}(\mathcal{I}(C, s_\alpha \lambda))$$

and

$$L^{-1} I_{s_\alpha}(\mathcal{I}(C, s_\alpha \lambda)) = 0.$$

This implies that

$$\Gamma(X, \mathcal{I}(O, \lambda)) = \Gamma(X, \mathcal{I}(C, s_\alpha \lambda)).$$

Since $s_\beta s_\alpha \lambda = -\lambda = \mu$ is antidominant, by the above discussion $\Gamma(X, \mathcal{I}(C, s_\alpha \lambda))$ is an idecomposable $\mathcal{U}_\theta$-module with V_μ as its unique irreducible submodule and $F_{\mu+\rho}$ as its unique irreducible quotient. Hence, $\Gamma(X, \mathcal{I}(O, \lambda)) = \tilde{I}_\mu$.

We can summarize this discussion in the following result.

1.4. Proposition. *Let λ be a weight and $\theta = W \cdot \lambda$. Let μ be the unique antidominant weight in θ.*

(i) *If λ is singular, there exists a unique irreducible Harish-Chandra module in $\mathcal{M}(\mathcal{U}_\theta, K)$. It is isomorphic to $\Gamma(X, \mathcal{I}(O, \lambda))$ and $\Gamma(X, \mathcal{I}(C, \lambda))$.*

(ii) *If λ is regular, there are two irreducible Harish-Chandra modules in $\mathcal{M}(\mathcal{U}_\theta, K)$. One of them is the finite dimensional module $F_{\mu+\rho}$ and the other is the infinite dimensional module V_μ.*

  (a) *If λ is antidominant, $\Gamma(X, \mathcal{I}(C, \lambda))$ is isomorphic to V_μ; $\Gamma(X, \mathcal{I}(O, \lambda))$ is isomorphic to I_μ.*

  (b) *If λ is dominant, $\Gamma(X, \mathcal{I}(O, \lambda))$ and $\Gamma(X, \mathcal{I}(C, \lambda))$ are isomorphic to $\tilde{I}_\mu$.*

  (c) *If λ is neither antidominant nor dominant, $\Gamma(X, \mathcal{I}(O, \lambda))$ is isomorphic to V_μ and $\Gamma(X, \mathcal{I}(C, \lambda))$ is isomorphic to $\tilde{I}_\mu$.*

Now we want to list the analogous results about the cohomology of the costandard sheaves $\mathcal{M}(Q, \lambda)$. Since they are obtained by a $\mathcal{D}$-module dualization process from the standard sheaves, they are equal to the standard Harish-Chandra sheaves on the closed orbit. On the open orbit, they differ from the corresponding standard sheaves if and only if the standard sheaves are reducible, i.e., if λ is a weight. Therefore, we may assume that λ is a weight. In this case, $\mathcal{O}(\lambda + \rho)$ is the unique irreducible quotient of $\mathcal{M}(O, \lambda)$ and $\mathcal{I}(C, \lambda)$ is the unique irreducible submodule; i.e., we have the following short exact sequence:

$$0 \to \mathcal{I}(C, \lambda) \to \mathcal{M}(O, \lambda) \to \mathcal{O}(\lambda + \rho) \to 0.$$

From the long exact sequence of cohomology, Borel-Weil-Bott theorem and 2, we conclude that

$$H^p(X, \mathcal{M}(O, \lambda)) = 0, \quad p > 0,$$

and

$$\Gamma(X, \mathcal{M}(O, \lambda)) = \Gamma(X, \mathcal{I}(C, \lambda))$$

for a singular weight λ. If λ is regular dominant weight, we have

$$\Gamma(X, \mathcal{M}(O, \lambda)) = \Gamma(X, \mathcal{I}(C, \lambda)), \quad H^1(X, \mathcal{M}(O, \lambda)) = H^1(X, \mathcal{I}(C, \lambda))$$

and

$$H^2(X, \mathcal{M}(O, \lambda)) = H^2(X, \mathcal{O}(\lambda + \rho)).$$

If λ is regular but neither dominant or antidominant, we have

$$\Gamma(X, \mathcal{M}(O, \lambda)) = \Gamma(X, \mathcal{I}(C, \lambda)), \quad H^1(X, \mathcal{M}(O, \lambda)) = H^1(X, \mathcal{O}(\lambda + \rho))$$

and

$$H^2(X, \mathcal{M}(O, \lambda)) = 0.$$

Finally, if λ is a regular antidominant weight, $H^p(X, \mathcal{M}(O, \lambda)) = 0$ for $p > 0$ and

$$0 \to \Gamma(X, \mathcal{I}(C, \lambda)) \to \Gamma(X, \mathcal{M}(O, \lambda)) \to \Gamma(X, \mathcal{O}(\lambda + \rho)) \to 0$$

is exact. Therefore, we have the following result.

1.5. Proposition. *Let λ be a weight and $\theta = W \cdot \lambda$. Let μ be the unique antidominant weight in θ.*

(i) *If λ is singular, $\Gamma(X, \mathcal{M}(O, \lambda))$ is the unique irreducible Harish-Chandra module in $\mathcal{M}(\mathcal{U}_\theta, K)$ and $H^p(X, \mathcal{M}(O, \lambda)) = 0$ for $p > 0$.*

(ii) *If λ is regular, we have:*

  (a) *if λ is antidominant, $\Gamma(X, \mathcal{M}(O, \lambda))$ is isomorphic to $\tilde{I}_\mu$ and $H^p(X, \mathcal{M}(O, \lambda)) = 0$ for $p > 0$;*

  (b) *if λ is dominant, $\Gamma(X, \mathcal{M}(O, \lambda))$ is isomorphic to $\tilde{I}_\mu$, and $H^1(X, \mathcal{M}(O, \lambda))$ and $H^2(X, \mathcal{M}(O, \lambda))$ are isomorphic to $F_{\mu+\rho}$.*

  (c) *if λ is neither dominant nor antidominant, $\Gamma(X, \mathcal{M}(O, \lambda))$ is isomorphic to $\tilde{I}_\mu$, $H^1(X, \mathcal{M}(O, \lambda))$ is isomorphic to $F_{\mu+\rho}$ and $H^2(X, \mathcal{M}(O, \lambda)) = 0$.*

Finally, we quote the following result about the action of the intertwining functors on costandard Harish-Chandra sheaves. We have

$$I_{s_\alpha}(\mathcal{M}(O, \lambda)) = \mathcal{M}(C, s_\alpha \lambda) \text{ and } L^{-1} I_{s_\alpha}(\mathcal{M}(O, \lambda)) = 0$$

and

$$I_{s_\beta}(\mathcal{M}(O, \lambda)) = \mathcal{M}(C, s_\beta \lambda) \text{ and } L^{-1} I_{s_\beta}(\mathcal{M}(O, \lambda)) = 0.$$

2. The geometric classification. The geometric classification of Harish-Chandra modules for $\mathrm{SL}(2,\mathbb{C})$ can be explained in the following picture. For any irreducible Harish-Chandra module in $\mathcal{M}(\mathcal{U}_\theta, K)$, there exists a unique $\mathrm{Re}\,\lambda$ which is antidominant for $\lambda \in \theta$. The possible $\mathrm{Re}\,\lambda$ are indicated by dotted lines in the negative chamber in the picture.

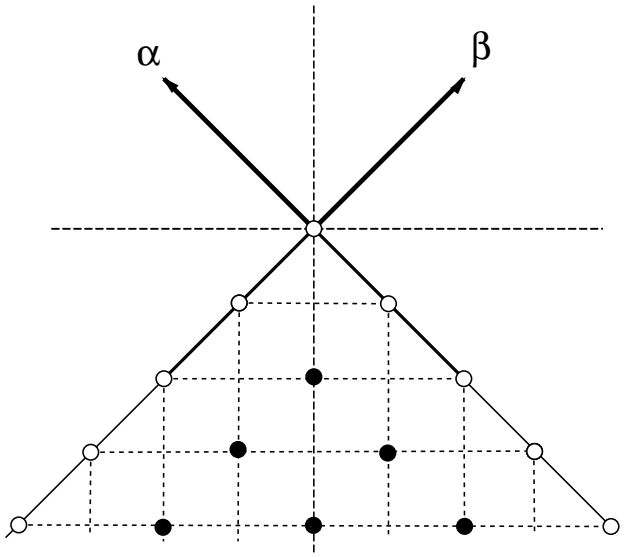

Figure 3. Geometric classification of irreducible Harish-Chandra modules

On horizontal dotted lines the standard Harish-Chandra sheaves exist for the closed orbit C and on the vertical dotted lines they exist for the open orbit.

If $\mathrm{Im}\,\lambda$ is nonzero, its kernel must be equal to $\mathfrak{t}_Q$, hence for such λ there is only one possible orbit Q.

Assume that θ is not a real orbit, i.e., that for strictly antidominant λ we have $\mathrm{Im}\,\lambda \neq 0$.

If $\mathrm{Re}\,\lambda$ is regular, such λ is unique, hence the orbit Q attached to an irreducible Harish-Chandra module V in $\mathcal{M}(\mathcal{U}_\theta, K)$ is uniquely determined by θ. Therefore, in this case there exists a unique irreducible Harish-Chandra module V in $\mathcal{M}(\mathcal{U}_\theta, K)$.

If $\mathrm{Re}\,\lambda$ is singular, there are two options. Either $\mathrm{Re}\,\lambda$ is orthogonal to one simple root or $\mathrm{Re}\,\lambda = 0$.

In the first case, let α be the simple root orthogonal to λ. Then both λ and $s_\alpha\lambda$ are antidominant. Using the results from §1 we see that the following holds. If there exists a standard Harish-Chandra sheaf in $\mathcal{M}(\mathcal{D}_\lambda, K)$ for the open orbit O, we have

$$\Gamma(X, \mathcal{I}(O,\lambda)) = \Gamma(X, \mathcal{I}(C, s_\alpha\lambda)).$$

If there exists a standard Harish-Chandra sheaf in $\mathcal{M}(\mathcal{D}_\lambda, K)$ for the closed orbit C, we have

$$\Gamma(X, \mathcal{I}(C,\lambda)) = \Gamma(X, \mathcal{I}(O, s_\alpha\lambda)).$$

Therefore, in both cases we see that there is a unique irreducible Harish-Chandra module in $\mathcal{M}(\mathcal{U}_\theta, K)$ which can be realized as the global sections of a standard module on either open or closed orbit (by choosing a different strictly antidominant λ).

If $\operatorname{Re}\lambda = 0$, the whole Weyl group orbit of λ is strictly antidominant. If there exists a standard Harish-Chandra sheaf for the open orbit O in $\mathcal{M}(\mathcal{D}_\lambda, K)$, we have

$$\Gamma(X, \mathcal{I}(O,\lambda)) = \Gamma(X, \mathcal{I}(O,-\lambda)) = \Gamma(X, \mathcal{I}(C, s_\alpha\lambda)) = \Gamma(X, \mathcal{I}(C, s_\beta\lambda)).$$

If there exists a standard Harish-Chandra sheaf for the closed orbit C in $\mathcal{M}(\mathcal{D}_\lambda, K)$, we have

$$\Gamma(X, \mathcal{I}(C,\lambda)) = \Gamma(X, \mathcal{I}(C,-\lambda)) = \Gamma(X, \mathcal{I}(O, s_\alpha\lambda)) = \Gamma(X, \mathcal{I}(O, s_\beta\lambda)).$$

Therefore, in both cases we see that there is a unique irreducible Harish-Chandra module in $\mathcal{M}(\mathcal{U}_\theta, K)$ which can be realized as global sections of a standard Harish-Chandra sheaf on either open or closed orbit (by choosing a different strictly antidominant λ).

If θ is real, we have $\operatorname{Re}\lambda = \lambda$ and it is uniquely determined by θ.

If λ is singular, it must be a weight, i.e., one of the white dots in our diagram. In this case $\Gamma(X, \mathcal{O}(\lambda+\rho)) = 0$, and the only irreducible Harish-Chandra module corresponds to the standard Harish-Chandra sheaf on the closed orbit C.

If λ is regular, it can be on one of the dotted lines or on their intersections. In the first case, there is only one infinite dimensional irreducible Harish-Chandra module in $\mathcal{M}(\mathcal{U}_\theta, K)$. It is equal to the global sections of the corresponding standard Harish-Chandra sheaf. In the second case, if λ is not a weight, there exist two irreducible infinite dimensional Harish-Chandra modules in $\mathcal{M}(\mathcal{U}_\theta, K)$, each of them corresponding to the irreducible standard Harish-Chandra sheaves for O and C. If λ is a weight, i.e., one of the black dots in the diagram, the irreducible Harish-Chandra modules in $\mathcal{M}(\mathcal{U}_\theta, K)$ are the irreducible finite dimensional representation $F_{\lambda+\rho}$ (attached to the open orbit) and the infinite dimensional Harish-Chandra module V_λ attached to the irreducible standard Harish-Chandra sheaf for C.

The Langlands invariant

$$\kappa_V = \tfrac{1}{2}(\operatorname{Re}\lambda - \sigma_Q \operatorname{Re}\lambda)$$

and Vogan-Zuckerman invariant

$$\kappa^V = \tfrac{1}{2}(\lambda + \sigma_Q\lambda)$$

of an irreducible Harish-Chandra module $V = \Gamma(X, \mathcal{L}(Q,\lambda))$ can be easily read off the Figures 1. and 2, depending on the K-orbit Q. Clearly,

$$\operatorname{Re}\lambda = \kappa_V + \kappa^V.$$

If $\operatorname{Re}\lambda$ is regular, it is uniquely determined by θ. If $\operatorname{Re}\lambda$ is singular with respect to a simple root γ, we have

$$s_\gamma(\kappa_V) = \kappa^V \text{ and } s_\gamma(\kappa^V) = \kappa_V.$$

Hence, if $\operatorname{Re}\lambda$ is singular, one these invariants can vanish if and only if $\operatorname{Re}\lambda = 0$, and in this case both of them are 0.

Consider first the vanishing of the Langlands invariant κ_V. By [2] this is equivalent with V being tempered. There are two options:

(i) $\operatorname{Re}\lambda$ is regular. In this case, we see from Figures 1. and 2, that $V = \Gamma(X, \mathcal{I}(C,\lambda)) = V_\lambda$ with $\operatorname{Re}\lambda$ one of the intersection points of the vertical axis with the horizontal dotted lines in Figure 2.

(ii) $\operatorname{Re}\lambda$ is singular. In this case $\kappa_V = \kappa^V = 0$, i.e., $\operatorname{Re}\lambda = 0$. If $\operatorname{Im}\lambda \neq 0$, its kernel is $\mathfrak{t}_Q$ and Q is determined by λ. Moreover, $\mathcal{I}(Q,\lambda)$ is irreducible, hence

$$V = \Gamma(X, \mathcal{I}(Q,\lambda)) = \Gamma(X, \mathcal{I}(Q,-\lambda)).$$

By replacing λ by $s_\alpha\lambda$ we switch Q with the other K-orbit in X. If $\operatorname{Im}\lambda = 0$, then

$$V = \Gamma(X, \mathcal{I}(O,0)) = \Gamma(X, \mathcal{I}(C,0)) = V_0.$$

This proves the following result.

2.1. Proposition. *Let V be a tempered irreducible Harish-Chandra module. Then θ is the orbit of $\lambda \in \mathfrak{h}^*$ such that*

$$\operatorname{Re}\lambda = -\tfrac{n}{4}(\alpha+\beta), n \in \mathbb{Z}_+;$$

and V is isomorphic to $\Gamma(X, \mathcal{I}(C,\lambda))$.

Conversely, for any such λ, the irreducible Harish-Chandra modules $\Gamma(X, \mathcal{I}(C,\lambda))$ are tempered. Moreover, the only nontrivial isomporphism is

$$\Gamma(X, \mathcal{I}(C,\lambda)) = \Gamma(X, \mathcal{I}(C,-\lambda))$$

for $\operatorname{Re}\lambda = 0$.

Now we consider the vanishing of the Vogan-Zuckerman invariant κ^V. Such Harish-Chandra modules V are called quasispherical in [2]. There are two options:

(i) $\operatorname{Re}\lambda$ is regular. In this case, we see from Figures 1. and 2, that $V = \Gamma(X, \mathcal{L}(O,\lambda))$ with $\operatorname{Re}\lambda$ one of the intersection points of the vertical axis with the horizontal dotted lines in Figure 1.

(ii) $\operatorname{Re}\lambda$ is singular. In this case $\kappa_V = \kappa^V = 0$. Hence V is tempered and considered already in the above discussion.

2.2. Proposition. *Let V be a quasispherical irreducible Harish-Chandra module. Then θ is the orbit of $\lambda \in \mathfrak{h}^*$ such that*

$$\operatorname{Re}\lambda = -c(\alpha+\beta), c \in \mathbb{R}_+;$$

and V is isomorphic to the unique irreducible submodule of $\Gamma(X, \mathcal{I}(O,\lambda))$.

Conversely, for any such λ, the unique irreducible submodules of $\Gamma(X, \mathcal{L}(O,\lambda))$ are quasispherical. Moreover, the only nontrivial isomporphism among them is

$$\Gamma(X, \mathcal{I}(O,\lambda)) = \Gamma(X, \mathcal{I}(O,-\lambda))$$

for $\operatorname{Re}\lambda = 0$.

3. The Langlands classification. If V is a nontempered irreducible Harish-Chandra module, i.e., its Langlands invariant $\kappa_V \neq 0$. Then V is a Langlands representation. Then, by Figures 1. and 2, it can correspond to an irreducible Harish-Chandra sheaf on the open orbit for $\operatorname{Re}\lambda \neq 0$ or to the irreducible standard sheaf on the closed orbit for $\operatorname{Re}\lambda$ away from the vertical axis. In the first case the Langlands invariant κ_V is in the negative chamber. In the second case, κ_V is in one of the chambers which are neither negative or positive, depending on which side of the vertical axis $\operatorname{Re}\lambda$ is lying. Hence, by applying a uniquely determined simple reflection s_α we can bring it to the negative chamber. The corresponding intertwining functor I_{s_α} establishes an isomorphism between $V \cong \Gamma(X, \mathcal{I}(C, \lambda))$ and $\Gamma(X, \mathcal{I}(O, s_\alpha\lambda))$ as we discussed in 1.

Putting everything together we get the well known picture of the Langlands classification for $\mathrm{SL}(2, \mathbb{C})$.

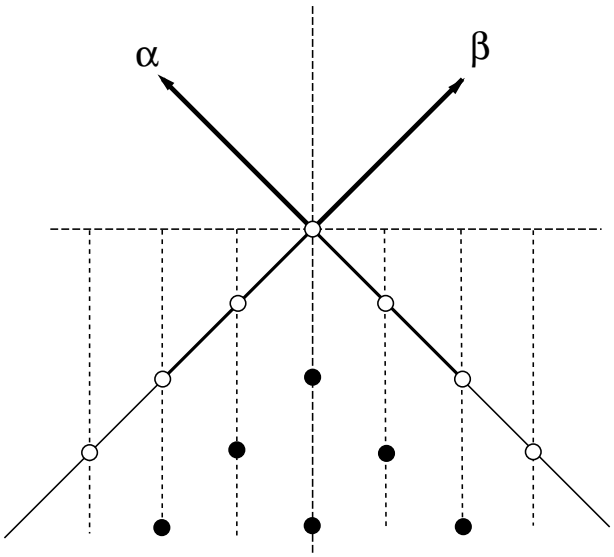

Figure 4. Langlands classification of irreducible Harish-Chandra modules

First, by the duality theorem of [1] one easily sees that the global sections of the standard Harish-Chandra sheaves $\mathcal{I}(O, \lambda)$ are principal series representations. Since the vertical axis corresponds to $\mathfrak{a}_O$, the global sections of the standard Harish-Chandra sheaves corresponding to $\operatorname{Re}\lambda$ strictly below the horizontal axis are the principal series in the negative Langlands chamber. The representations reduce at the black dots and have irreducible finite dimensional representations as unique irreducible submodules at these points.

4. The Vogan-Zuckerman classification. Combining 2.2 and §3, we see that all quasispherical representations of $\mathrm{SL}(2, \mathbb{C})$ are either the irreducible spherical principal series or the irreducible finite dimensional spherical representations. The rest of the irreducible representations with $\kappa^V \neq 0$ correspond either to the standard modules on the closed orbit for $\operatorname{Re}\lambda \neq 0$ or to the irreducible modules attached to the open orbit for $\operatorname{Re}\lambda$ off the vertical axis. In the second case, κ^V is in one of the chambers which are neither negative or positive, depending on which side of the vertical axis $\operatorname{Re}\lambda$ is lying. Hence, by applying a uniquely determined simple reflection s_α we can bring it to the negative chamber. The

corresponding intertwining functor I_{s_α} establishes an isomorphism between $\Gamma(X, \mathcal{M}(O, \lambda))$ and $\Gamma(X, \mathcal{M}(C, s_\alpha\lambda))$ as we discussed in 1. Hence, we see that the latter modules V are isomorphic to the unique irreducible quotients of the global sections of the costandard modules on the closed orbit for the parameter $s_\alpha\lambda$. Therefore, the non-quasispherical representations correspond to the unique irreducible quotients of $\Gamma(X, \mathcal{M}(C, \mu)) = \Gamma(X, \mathcal{I}(C, \mu))$ for $\operatorname{Re}\mu$ strictly below the horizontal axis in the following picture.

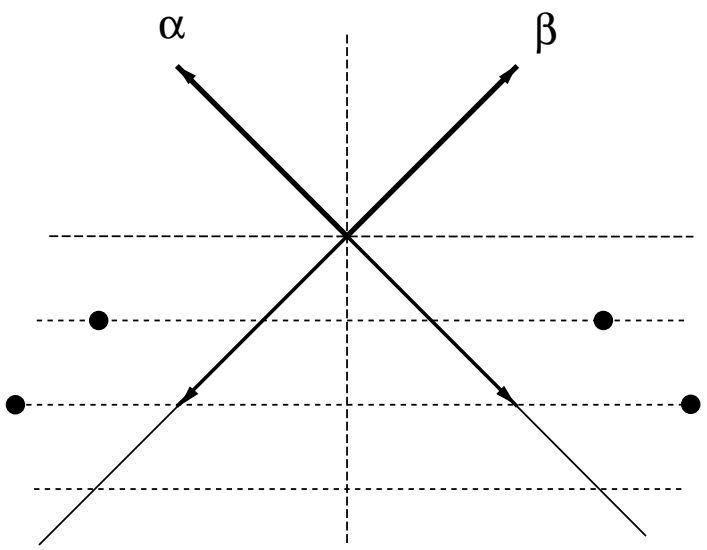

Figure 5. Vogan-Zuckerman classification of irreducible Harish-Chandra modules

Again, by the duality theorem of [1], we can see that these representations are the standard Zuckerman modules obtained by the "cohomological induction".

The black dots correspond to the weights where the modules $\Gamma(X, \mathcal{I}(C, \lambda))$ are reducible.

References

1. H. Hecht, D. Miličić, W. Schmid, J. A. Wolf, *Localization and standard modules for real semisimple Lie groups I: The duality theorem*, Inventiones Math. **90** (1987), 297–332.
2. H. Hecht, D. Miličić, W. Schmid, J. A. Wolf, *Localization and standard modules for real semisimple Lie groups II: Irreducibility, vanishing theorems and classification*, (in preparation).
3. D. Miličić, *Algebraic $\mathcal{D}$-modules and representation theory of semisimple Lie groups*, Analytic Cohomology and Penrose Transform (M. Eastwood, J.A. Wolf, R. Zierau, eds.), Contemporary Mathematics, vol. 154, 1993, pp. 133–168.

Department of Mathematics, University of Utah, Salt Lake City, Utah 84112
E-mail address: milicic@math.utah.edu